Math 309 - Spring-Summer 2017

Solutions to Problem Set # 6

Completion Date: Friday June 16, 2017

Question 1.

With the aid of expressions

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

and

$$|\cos z|^2 = \cos^2 x + \sinh^2 y,$$

show that

(a) $|\sinh y| \le |\sin z| \le \cosh y$;

(b) $|\sinh y| \le |\cos z| \le \cosh y$.

Solution:

(a) Note that

$$\begin{aligned} |\sin z|^2 &= |\sin(x + i\,y)|^2 \\ &= |\sin x\ \cos(iy) + \cos x\ \sin(iy)|^2 \\ &= |\sin x\ \cosh y + i\cos x\ \sinh y|^2 \\ &= \sin^2 x\ \cosh^2 y + \cos^2 x\ \sinh^2 y \\ &\le \sin^2 x\ \cosh^2 y + \cos^2 x\ \cosh^2 y \\ &= \cosh^2 y, \end{aligned}$$

since $\sinh^2 y \le \cosh^2 y$ for all $y \in \mathbb{R}$, and $|\sin z| \le \cosh y$.

Also,

$$\begin{aligned} |\sin z|^2 &= \sin^2 x\ \cosh^2 y + \sinh^2 y\ \cos^2 x \\ &\ge \sin^2 x\ \sinh^2 y + \cos^2 x\ \sinh^2 y \\ &= \sinh^2 y, \end{aligned}$$

and $|\sinh y| \le |\sin z|$.

(b) Note that

$$\begin{aligned} |\cos z|^2 &= |\cos(x + i\,y)|^2 \\ &= |\cos x\ \cos(iy) - \sin x\ \sin(iy)|^2 \\ &= |\cos x\ \cosh y - i\sin x\ \sinh y|^2 \\ &= \cos^2 x\ \cosh^2 y + \sin^2 x\ \sinh^2 y \\ &\le \cos^2 x\ \cosh^2 y + \sin^2 x\ \cosh^2 y \\ &= \cosh^2 y, \end{aligned}$$

and $|\cos z| \le \cosh y$.

Also,

$$\begin{aligned}|\cos z|^2 &= \cos^2 x\,\cosh^2 y + \sin^2 x\,\sinh^2 y\\ &\geq \cos^2 x\,\sinh^2 y + \sin^2 x\,\sinh^2 y\\ &= \sinh^2 y\end{aligned}$$

and $|\cos z| \geq |\sinh y|$.

Question 2.

Show that

(a) $\overline{\cos(i\,z)} = \cos(i\,\overline{z})$ for all z;

(b) $\overline{\sin(i\,z)} = \sin(i\,\overline{z})$ if and only if $z = n\pi i \quad (n = 0, \pm1, \pm2, \dots)$.

SOLUTION:

(a) If $z = x + i\,y \in \mathbb{C}$, then

$$\cos(i\,\overline{z}) = \cos(y + i\,x) = \cosh x\,\cos y - i\,\sin y\,\sinh x$$

and

$$\cos(i\,z) = \cos(-y + i\,x) = \cosh x\,\cos(-y) - i\,\sin(-y)\,\sinh x$$

that is,

$$\cos(i\,z) = \cosh x\,\cos y + i\,\sin y\,\sinh x,$$

and $\overline{\cos(i\,z)} = \cos(i\,\overline{z})$ for all $z \in \mathbb{C}$.

(b) Since

$$\sin(i\,\overline{z}) = \sin y\,\cosh x + i\,\cos y\,\sinh x$$

and

$$\overline{\sin(i\,z)} = -\sin y\,\cosh x - i\,\cos y\,\sinh x$$

then $\sin(i\,\overline{z}) = \overline{\sin(i\,z)}$ if and only if

$$\begin{aligned}2\sin y\,\cosh x &= 0\\ 2\cos y\,\sinh x &= 0.\end{aligned}$$

Now, since $\cosh x \geq 1$, the first of these equations holds if and only if $\sin y = 0$, and then in the second equation since $\cos y \neq 0$, we must have $\sinh x = 0$, therefore these two equations hold if and only if

$$x = 0 \qquad \text{and} \qquad y = n\pi, \quad \text{for} \quad n = 0, \pm1, \pm2, \dots$$

and so

$$\sin(i\,\overline{z}) = \overline{\sin(i\,z)}$$

if and only if $z = n\pi i$, for $n = 0, \pm1, \pm2, \dots$.

Question 3.

Find all roots of the equation $\sin z = \cosh 4$ by equating real and imaginary parts of $\sin z$ and $\cosh 4$.

Ans: $\left(\dfrac{\pi}{2} + 2n\pi\right) \pm 4i \quad (n = 0, \pm1, \pm2, \dots)$.

SOLUTION: Note that

$$\sin z = \sin(x + i\,y) = \sin x\,\cosh y + i\,\cos x\,\sinh y = \cosh 4$$

if and only if

$$\begin{aligned} \sin x\,\cosh y &= \cosh 4 \\ \sinh y\,\cos x &= 0. \end{aligned}$$

Now, if $\sinh y = 0$, then $\cosh y = 1$, and the first equation implies that

$$\sin x = \cosh 4 > 1$$

which is a contradiction. Therefore we must have $\sinh y \neq 0$, and the second equation implies that $\cos x = 0$, so that

$$x = \frac{(2n+1)\pi}{2}$$

for $n = 0, \pm 1, \pm 2, \ldots$. For these values of x we have $\sin x = \pm 1$, and since $\cosh 4 > 0$, and $\cosh y > 0$, then we must have $\sin x = +1$, and $\cosh y = \cosh 4$, so that

$$y = \pm 4 \qquad \text{and} \qquad x = \frac{(4n+1)\pi}{2}$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Therefore, the solutions to the equation $\sin z = \cosh 4$ are

$$z = \frac{(4n+1)\pi}{2} \pm 4i,$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 4.

Show that $|\sinh x| \leq |\cosh z| \leq \cosh x$ by using

(a) the identity $|\cosh z|^2 = \sinh^2 x + \cos^2 y$;

(b) the inequalities $|\sinh y| \leq |\cos z| \leq \cosh y$.

SOLUTION:

(a) We have

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y \geq \sinh^2 x \tag{1}$$

and

$$|\cosh z|^2 = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y \leq \cosh^2 x \cos^2 y + \cosh^2 x \sin^2 y,$$

that is,

$$|\cosh z|^2 \leq \cosh^2 x \tag{2}$$

and combining (1) and (2) we get $|\sinh x| \leq |\cosh z| \leq \cosh x$.

(b) Starting from the inequality

$$|\sinh y| \leq |\cos z| \leq \cosh y,$$

we replace z by iz, then since

$$iz = -y + i\,x \qquad \text{and} \qquad \cos(iz) = \cosh z,$$

we have

$$|\sinh(\mathrm{Im}(iz))| \leq |\cos(iz)| \leq \cosh(\mathrm{Im}(iz)),$$

that is,

$$|\sinh x| \leq |\cosh z| \leq \cosh x.$$

Question 5.

Locate all zeros and singularities of the hyperbolic tangent function.

Solution: Note that

$$\tanh z = \frac{\sinh z}{\cosh z} = 0 \quad \text{if and only if} \quad \sinh z = 0 \quad \text{if and only if} \quad e^z = e^{-z} \quad \text{if and only if} \quad e^{2z} = 1,$$

that is,

$$\tanh z = 0 \quad \text{if and only if} \quad e^{2x} \cdot e^{2iy} = 1 \quad \text{if and only if} \quad e^{2x} = 1 \quad \text{and} \quad 2y = 2\pi n$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Therefore $\tanh z = 0$ if and only if $z = n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots$.

Note that the singularities of $\tanh z$ are precisely the points $z \in \mathbb{C}$ for which $\cosh z = 0$, and

$$\cosh z = 0 \quad \text{if and only if} \quad e^z = -e^{-z} \quad \text{if and only if} \quad e^{2z} = -1,$$

that is,

$$\cosh z = 0 \quad \text{if and only if} \quad e^{2x} \cdot e^{2iy} = -1 = e^{\pi i} \quad \text{if and only if} \quad e^{2x} = 1 \quad \text{and} \quad 2y = \pi + 2\pi n$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Therefore $\cosh z = 0$ if and only if

$$z = \frac{\pi i}{2} + n\pi i = \left(n + \frac{1}{2}\right)\pi i$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 6.

Find all roots of the equation $\cosh z = -2$.

Ans: $\pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

Solution: Note that

$$\cosh z = \cosh x \, \cos y + i \, \sinh x \, \sin y = -2$$

if and only if

$$\begin{aligned} \cosh x \, \cos y &= -2 \\ \sinh x \, \sin y &= 0 \end{aligned}$$

Now, if $\sinh x = 0$, then $x = 0$ and this implies that $\cosh x = 1$, and then the first equation implies that $\cos y = -2$ which is a contradiction. Therefore, $\sinh x \neq 0$, and from the second equation we must have $\sin y = 0$. Thus, y is a multiple of π, and since $\cosh x \geq 1$, then we must have $\cos y = -1$, and $\cosh x = 2$.

Therefore, $\cosh z = -2$ if and only if

$$x = \cosh^{-1}(2), \qquad \text{and} \qquad y = (2n + 1)\pi$$

for $n = 0, \pm 1, \pm 2, \ldots$, that is, if and only if

$$z = \cosh^{-1}(2) + (2n + 1)\pi i$$

for $n = 0, \pm 1, \pm 2, \ldots$.

In order to simplify the expression for $\cosh^{-1}(2)$, note that $x = \cosh^{-1}(2)$ if and only if

$$\cosh x = \frac{e^x + e^{-x}}{2} = 2,$$

that is, if and only if

$$e^{2x} - 4e^{x} + 1 = 0,$$

and solving this quadratic equation, we get two real roots,

$$e^{x} = 2 \pm \sqrt{3}$$

or

$$x = \ln\left(2 \pm \sqrt{3}\right).$$

However, note that

$$\ln\left(2 - \sqrt{3}\right) = \ln\left(\frac{(2-\sqrt{3})(2+\sqrt{3})}{2+\sqrt{3}}\right) = \ln\left(\frac{1}{2+\sqrt{3}}\right) = -\ln\left(2+\sqrt{3}\right),$$

and we have $\cosh z = -2$ if and only if

$$z = \pm\ln\left(2+\sqrt{3}\right) + (2n+1)\pi i$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 7.

Solve the equation $\sin z = 2$ for z by

(a) equating real and imaginary parts in that equation;

(b) Using expression $\sin^{-1} z = -i \log\left[i\, z + (1 - z^2)^{1/2}\right]$.

SOLUTION:

(a) We have

$$\sin z = \sin(x + i\, y) = \sin x\, \cosh y + i\, \cos x\, \sinh y = 2$$

if and only if

$$\begin{aligned} \sin x\, \cosh y &= 2 \\ \cos x\, \sinh y &= 0. \end{aligned}$$

If these equations hold and $\sinh y = 0$, then $y = 0$ and so $\cosh y = 1$, and from the first equation this implies that $\sin x = 2$, which is a contradiction. Therefore, $\sinh y \neq 0$, and from the second equation we must have $\cos x = 0$, so that $x = \dfrac{(2n+1)\pi}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$. Also, since $\cosh y \geq 1 > 0$, then $\sin x = +1$ and $\cosh y = 2$.

Therefore, $\sin z = 2$ if and only if

$$x = \frac{(2n+1)\pi}{2}, \quad \text{where} \quad n \quad \text{is an even integer,} \qquad \text{and} \qquad y = \pm\ln\left(2+\sqrt{3}\right),$$

that is, if and only if

$$z = (4n+1)\frac{\pi}{2} \pm i\, \ln\left(2+\sqrt{3}\right)$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(b) Using

$$\sin^{-1} z = -i\, \log\left[iz + (1 - z^2)^{1/2}\right]$$

with $z = 2$, we get

$$\begin{aligned}\sin^{-1}(2) &= -i \,\log\left[2i + (-3)^{1/2}\right] \\ &= -i \,\log[\,2i \pm \sqrt{3}\,i\,] \\ &= -i \,\log[\,i(2 \pm \sqrt{3})\,]\end{aligned}$$

and by definition of the logarithm,

$$\log\, i(2+\sqrt{3}) = \log\left[(2+\sqrt{3})e^{i\pi/2}\right] = \ln(2+\sqrt{3}) + i\left[\frac{\pi}{2} + 2\pi n\right]$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Since

$$2 - \sqrt{3} = \frac{1}{2+\sqrt{3}},$$

the roots of the equation $\sin z = 2$ are given by

$$z = \sin^{-1} 2 = -i\left(\pm \ln(2+\sqrt{3}) + i\left(\frac{\pi}{2} + 2\pi n\right)\right),$$

that is,

$$z = \sin^{-1} 2 = \frac{(4n+1)\pi}{2} \pm i\,\ln\left(2+\sqrt{3}\right)$$

for $n = 0, \pm 1, \pm 2, \ldots$, as before.