Math 309 - Spring-Summer 2017

Solutions to Problem Set # 5

Completion Date: Friday June 9, 2017

Question 1.

Show that $\text{Log}(1-i) = \dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}\,i$.

SOLUTION: Since

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2}e^{-i\pi/4},$$

then $\text{Arg}(1-i) = -\dfrac{\pi}{4}$, and

$$\text{Log}(1-i) = \ln\sqrt{2} + i\,\text{Arg}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}\,i.$$

Question 2.

Verify that when $n = 0, \pm 1, \pm 2, \ldots,\quad \log i = \left(2n + \dfrac{1}{2}\right)\pi\, i.$

SOLUTION: Since

$$i = 0 + i\,1 = 1 \cdot e^{i\pi/2} = 1 \cdot e^{i(\pi/2 + 2\pi n)},$$

for $n = 0, \pm 1, \pm 2, \ldots$ then

$$|i| = 1 \qquad \text{and} \qquad \arg(i) = \frac{\pi}{2} + 2\pi n$$

for $n = 0, \pm 1, \pm 2, \ldots$, and therefore

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) = \left(2n + \frac{1}{2}\right)\pi i,$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 3.

Verify that when $n = 0, \pm 1, \pm 2, \ldots,\quad \log\left(-1 + \sqrt{3}\,i\right) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi\, i.$

SOLUTION: Since

$$-1 + \sqrt{3}\,i = 2\left(\frac{-1}{2} + \frac{\sqrt{3}\,i}{2}\right) = 2 \cdot e^{2\pi i/3},$$

then

$$|-1 + \sqrt{3}\,i| = 2 \qquad \text{and} \qquad \arg(-1 + \sqrt{3}\,i) = \frac{2\pi}{3} + 2\pi n$$

for $n = 0, \pm 1, \pm 2, \ldots$, and therefore

$$\log(-1 + \sqrt{3}\,i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 4.

Show that

(a) $\operatorname{Log}(1+i)^2 = 2\operatorname{Log}(1+i)$; (b) $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$.

SOLUTION:

(a) Note that

$$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\cdot e^{i\pi/4},$$

while

$$(1+i)^2 = 2\cdot i = 2\cdot e^{i\pi/2},$$

and therefore

$$\operatorname{Arg}(1+i) = \frac{\pi}{4} \qquad \text{and} \qquad \operatorname{Arg}(1+i)^2 = \frac{\pi}{2},$$

so that

$$\operatorname{Log}(1+i)^2 = \ln 2 + \frac{\pi i}{2} = 2\left(\ln\sqrt{2} + \frac{\pi i}{4}\right) = 2\cdot\operatorname{Log}(1+i).$$

(b) Note that

$$-1+i = \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\cdot e^{3i\pi/4},$$

while

$$(-1+i)^2 = -2\cdot i = 2(-i) = 2\cdot e^{-\pi i/2},$$

and therefore

$$\operatorname{Arg}(-1+i) = \frac{3\pi}{4} \qquad \text{and} \qquad \operatorname{Arg}(-1+i)^2 = -\frac{\pi}{2},$$

so that

$$\operatorname{Log}(-1+i) = \ln\sqrt{2} + \frac{3\pi i}{4} \qquad \text{and} \qquad \operatorname{Log}(-1+i)^2 = \ln 2 - \frac{\pi i}{2},$$

and

$$2\cdot\operatorname{Log}(-1+i) = \ln 2 + \frac{3\pi i}{2} \neq \ln 2 - \frac{\pi i}{2} = \operatorname{Log}(-1+i)^2.$$

Question 5.

Show that

(a) the set of values of $\log\left(i^{1/2}\right)$ is $\left(n+\frac{1}{4}\right)\pi i \quad (n = 0, \pm1, \pm2, \ldots)$ and that the same is true of $\frac{1}{2}\log i$.

(b) the set of values of $\log\left(i^2\right)$ is *not* the same as the set of values of $2\log i$.

SOLUTION:

(a) We have

$$\log\left(i^{1/2}\right) = \log\left(e^{\frac{\log i}{2}}\right) = \log\left(e^{\frac{1}{2}(\ln|i| + i\arg(i))}\right) = \log\left(e^{\frac{1}{2}(0+i(\pi/2+2\pi n))}\right)$$

that is,

$$\log\left(i^{1/2}\right) = \log\left(e^{i(\pi/4+\pi n)}\right) = i\left(\frac{\pi}{4}+\pi n\right) + i\,2\pi m,$$

and therefore

$$\log\left(i^{1/2}\right) = i\left(\frac{\pi}{4} + p\pi\right)$$

for $p = 0, \pm 1, \pm 2, \ldots$ since $(n+2m)\pi$ can take the values

$$0 \cdot \pi,\ \pm\pi,\ \pm 2\pi,\ \ldots$$

as m and n run through the integers.

Also,

$$\tfrac{1}{2}\log(i) = \frac{1}{2}\left[\ln|i| + i\left(\frac{\pi}{2} + 2\pi k\right)\right]$$

for $k = 0, \pm 1, \pm 2, \ldots$, that is,

$$\tfrac{1}{2}\log(i) = i\left(\frac{\pi}{4} + k\pi\right)$$

for $k = 0, \pm 1, \pm 2, \ldots$.

Therefore, the set of values of $\log\left(i^{1/2}\right)$ is the same as the set of values of $\frac{1}{2}\log i$.

(b) Note that

$$\log(i^2) = \log\left(e^{2\log i}\right) = \log\left(e^{2(0+i(\pi/2+2\pi n))}\right) = \log e^{i(\pi+4\pi n)} = i(\pi + 4\pi n) + i\,2\pi m$$

for $m = 0, \pm 1, \pm 2, \ldots$ and $n = 0, \pm 1, \pm 2, \ldots$, so that

$$\log(i^2) = i(\pi + 2k\pi)$$

for $k = 0, \pm 1, \pm 2, \ldots$.

Also,

$$2\log(i) = 2\left(\ln|i| + i\arg(i)\right) = 2\left[0 + i\left(\frac{\pi}{2} + 2\pi n\right)\right] = i(\pi + 4\pi n)$$

for $n = 0, \pm 1, \pm 2, \ldots$, and therefore

$$\log(i^2) \neq 2\log(i),$$

that is, the values are different, since $\{4\pi n : n \in \mathbb{Z}\}$ is different from $\{2\pi k : k \in \mathbb{Z}\}$.

Question 6.

Show that

(a) the function $\mathrm{Log}(z-i)$ is analytic everywhere except on the half line $y = 1$ $(x \leq 0)$;

(b) the function

$$\frac{\mathrm{Log}(z+4)}{z^2+i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

Solution:

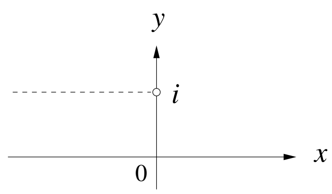
(a) Let $f(z) = \mathrm{Log}(z-i)$, then f is analytic at z provided that $w = z - i$ satisfies

$$|w| > 0 \qquad \text{and} \qquad -\pi < \mathrm{Arg}\ w < \pi.$$

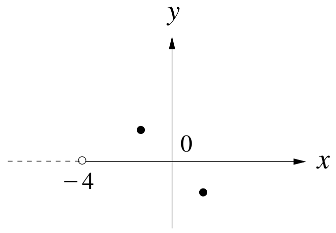
Writing $w = \rho e^{i\phi}$ where $\rho > 0$ and $-\pi < \phi < \pi$, then the set of all $z \in \mathbb{C}$ such that

$$z = i + w = i + \rho e^{i\phi},$$

where $\rho > 0$ and $-\pi < \phi < \pi$ describes the set of all points in the z-plane except $z = i$ and the half-line $y = 1,\ x < 0$.



(b) Let $f(z) = \dfrac{\operatorname{Log}(z+4)}{z^2+i}$, then f is analytic at all points except $z = \pm\dfrac{(1-i)}{\sqrt{2}}$ (the square roots of $-i$) and at points where $\operatorname{Log}(z+4)$ is not analytic, that is, at $z = -4$ and points on the half-line $y = 0,\ x < -4$.



Question 7.

Show that when $n = 0, \pm 1, \pm 2, \ldots$,

(a) $(1+i)^i = \exp\left(-\dfrac{\pi}{4} + 2n\pi\right)\exp\left(\dfrac{i}{2}\ln 2\right)$;

(b) $(-1)^{1/\pi} = e^{(2n+1)i}$.

Solution:

(a) Note that

$$(1+i)^i = e^{i\log(1+i)} = e^{i\left[\ln|1+i| + i\left(\frac{\pi}{4}+2\pi n\right)\right]},$$

so that

$$(1+i)^i = e^{\left[\frac{i}{2}\ln 2 - \left(\frac{\pi}{4}+2\pi n\right)\right]},$$

that is,

$$(1+i)^i = e^{-\left(\frac{\pi}{4}+2\pi n\right)} \cdot \left(\cos(\ln\sqrt{2}) + i\,\sin(\ln\sqrt{2})\right)$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(b) Note that

$$(-1)^{1/\pi} = e^{\frac{1}{\pi}\log(-1)} = e^{\frac{1}{\pi}(\ln|-1| + i(\pi+2\pi n))},$$

that is,

$$(-1)^{1/\pi} = e^{\frac{1}{\pi}(2n+1)\pi i} = e^{(2n+1)i}$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Question 8.

Find the principal value of $(1-i)^{4i}$.

Solution: Note that

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \sqrt{2}e^{-i\pi/4},$$

and $\operatorname{Arg}(1-i) = -\dfrac{\pi}{4}$.

Therefore,

$$(1-i)^{4i} = e^{4i\log(1-i)} = e^{4i\left[\ln|1-i| + i\left(-\frac{\pi}{4}+2\pi n\right)\right]}$$

for $n = 0, \pm 1, \pm 2, \ldots$, that is,

$$(1-i)^{4i} = e^{4i\left(\ln \sqrt{2}\right)} \cdot e^{-(-\pi + 8\pi n)}$$

for $n = 0, \pm 1, \pm 2, \ldots$, that is,

$$(1-i)^{4i} = e^{\pi(1-8n)} \cdot \left(\cos(4 \ln \sqrt{2}) + i \sin(4 \ln \sqrt{2})\right)$$

for $n = 0, \pm 1, \pm 2, \ldots$.

The principal value of $(1-i)^{4i}$ occurs when $n = 0$, and has the value

$$(1-i)^{4i} = e^{\pi} \left(\cos(4 \ln \sqrt{2}) + i \sin(4 \ln \sqrt{2})\right).$$