Math 309 - Spring-Summer 2017

Solutions to Problem Set # 4

Completion Date: Friday June 2, 2017

Question 1.

Verify that the function

$$f(z) = 3x + y + i\,(3y - x)$$

is entire.

SOLUTION: If $f(z) = 3x + y + i(3y - x)$, then $u(x, y) = 3x + y$ and $v(x, y) = 3y - x$, so that

$$\frac{\partial u}{\partial x} = 3 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 1 = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations hold for all $z \in \mathbb{C}$ and all partial derivatives are continuous everywhere, $f'(z)$ exists for all $z \in \mathbb{C}$ and $f(z)$ is analytic at each $z \in \mathbb{C}$. Therefore $f(z)$ is an entire function.

Note that $f(z) = 3(x + i\,y) + i(-x - i\,y) = 3z - i\,z$ and $f'(z) = 3 - i$.

Question 2.

Verify that the function

$$f(z) = e^{-y} \sin x - i\,e^{-y} \cos x$$

is entire.

SOLUTION: If $f(z) = e^{-y} \sin x - i\,e^{-y} \cos x$, then $u(x, y) = e^{-y} \sin x$ and $v(x, y) = -e^{-y} \cos x$, so that

$$\frac{\partial u}{\partial x} = e^{-y} \cos x = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -e^{-y} \sin x = -\frac{\partial v}{\partial x}.$$

Since all partial derivatives are defined and continuous everywhere and the Cauchy-Riemann equations hold for all $z \in \mathbb{C}$, then $f'(z)$ exists for all $z \in \mathbb{C}$, that is, f is entire.

Question 3.

For the function

$$f(z) = \frac{z^2 + 1}{(z + 2)(z^2 + 2z + 2)},$$

determine the singular points of the function and state why the function is analytic everywhere except at those points.

Ans: $z = -2,\ -1 \pm i$.

SOLUTION: Note that $z^2 + 2z + 2 = (z + 1)^2 + 1 = 0$ when $z = -1 \pm i$, and $f'(z)$ doesn't exist for

$$z_0 = -2, \quad z_1 = -1 + i, \quad z_2 = -1 - i,$$

and f is not analytic at any of these points.

Note that $f'(z)$ exists except at each of these points, so that f is analytic everywhere except at these points.

Therefore, given any one of these points, every ϵ-neighborhood of that point contains at least one point at which f is analytic, and the points

$$z_0 = -2, \quad z_1 = -1 + i, \quad z_2 = -1 - i,$$

are singular points of $f(z)$.

Question 4.

Verify that the function
$$g(z) = \ln r + i\,\theta \quad (r > 0,\ 0 < \theta < 2\pi)$$
is analytic in the indicated domain of definition, with derivative $g'(z) = \dfrac{1}{z}$. Then show that the composite function $G(z) = g(z^2 + 1)$ is analytic in the quadrant $x > 0,\ y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2 + 1}.$$

Suggestion: Observe that $\mathrm{Im}(z^2 + 1) > 0$ when $x > 0,\ y > 0$.

Solution: If $g(z) = \ln r + i\,\theta \quad (r > 0,\ 0 < \theta < 2\pi)$, then

$$u(r,\theta) = \ln r \qquad \text{and} \qquad v(r,\theta) = \theta$$

for $r > 0,\ 0 < \theta < 2\pi$, and

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} &= 0 = -\frac{\partial v}{\partial r}. \end{aligned}$$

Therefore, the Cauchy-Riemann equations hold at each point of the domain $r > 0,\ 0 < \theta < 2\pi$, and all the partial derivatives are continuous there, hence g is analytic in this domain, and

$$g'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{1}{r} + i\,0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z},$$

for $r > 0,\ 0 < \theta < 2\pi$.

Since the function $z \mapsto z^2 + 1$ is analytic everywhere, then the composition $G(z) = g(z^2 + 1)$ is analytic wherever
$$w = z^2 + 1 = \rho e^{i\phi}$$
satisfies $\rho > 0,\ 0 < \phi < 2\pi$.

Now, if $x > 0$ and $y > 0$, then $\mathrm{Im}(z^2 + 1) = 2xy > 0$, so that $\rho > 0$ and $0 < \phi < \pi < 2\pi$, and therefore
$$G(z) = g(z^2 + 1)$$
is analytic for $z = x + i\,y$ with $x > 0,\ y > 0$, and we can use the chain rule to differentiate it. Letting $w = z^2 + 1$, then
$$G'(z) = g'(w)\cdot\frac{dw}{dz} = \frac{1}{w}\cdot 2z = \frac{2z}{z^2 + 1},$$

for $x > 0,\ y > 0$.

Question 5.

Let $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant throughout D if $|f(z)|$ is constant throughout D.

Suggestion: Observe that

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{if} \quad |f(z)| = c \ (c \neq 0).$$

SOLUTION: Suppose that $f(z) = u(x,y) + i\,v(x,y)$ for $z = x + i\,y \in D$, then

$$|f(z)|^2 = u(x,y)^2 + v(x,y)^2 = c^2$$

for $z \in D$, where $c \in \mathbb{R}$ is a constant. Differentiating with respect to x and with respect to y, we get

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} &= 0 \\ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} &= 0 \end{aligned}$$

in D. Multiplying the first equation by u and the second equation by v, we get

$$\begin{aligned} u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} &= 0 \\ uv\frac{\partial u}{\partial y} + v^2\frac{\partial v}{\partial y} &= 0, \end{aligned}$$

and using the Cauchy-Riemann equations, we have

$$\begin{aligned} u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} &= 0 \\ -uv\frac{\partial v}{\partial x} + v^2\frac{\partial u}{\partial x} &= 0. \end{aligned}$$

Adding these two equations, we have

$$(u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

on D. In an entirely similar way, we find

$$(u^2 + v^2)\frac{\partial u}{\partial y} = 0, \quad \text{and} \quad (u^2 + v^2)\frac{\partial v}{\partial x} = 0, \quad \text{and} \quad (u^2 + v^2)\frac{\partial v}{\partial y} = 0,$$

on D. Finally, since $u^2 + v^2 = c^2$, we have

$$c^2\frac{\partial u}{\partial x} = c^2\frac{\partial u}{\partial y} = c^2\frac{\partial v}{\partial x} = c^2\frac{\partial v}{\partial y} = 0$$

on D. Now, if $c = 0$, then $|f(z)| = 0$ and therefore $f(z) = 0$ for all $z \in D$, however, if $c \neq 0$, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

and therefore $f(z)$ is a constant for $z \in D$.

Alternatively, using the suggestion, if $|f(z)| = c$ for all $z \in D$, and $c = 0$, then $f(z) = 0$ for all $z \in D$.

On the other hand, if $|f(z)| = c$ for all $z \in D$, where $c \neq 0$, then $f(z)$ is never 0 in D, and the function

$$\overline{f(z)} = \frac{c^2}{f(z)}$$

is also analytic on D, and since f and $\overline{f}$ are both analytic on D, then $f(z)$ is a constant on D.

Question 6. Show that the function
$$u(x, y) = \sinh x \sin y$$
is harmonic in some domain and find a harmonic conjugate $v(x, y)$.

Ans: $v(x, y) = -\cosh x \cos y$.

SOLUTION: If $u(x, y) = \sinh x\, \sin y$, then
$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = \sinh x\, \sin y - \sinh x\, \sin y = 0$$
for all $(x, y) \in \mathbb{C}$, and u is harmonic on all of $\mathbb{C}$. In order to find a harmonic conjugate $v(x, y)$ of u, we use the Cauchy-Riemann equations to get
$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \cosh x\, \sin y$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\sinh x\, \cos y,$$
Integrating the first equation with respect to y holding x fixed, we have
$$v(x, y) = -\cosh x\, \cos y + h(x)$$
where $h(x)$ is an arbitrary function of x. Using the second equation, we have
$$\frac{\partial v}{\partial x} = -\sinh x\, \cos y + h'(x) = -\sinh x\, \cos y,$$
that is, $h'(x) = 0$ for all x, so that $h(x) = C$ (constant) for all x.

Therefore, for any real constant C, the function
$$v(x, y) = -\cosh x\, \cos y + C$$
is a harmonic conjugate for $u(x, y) = \sinh x\, \sin y$ on $\mathbb{C}$.

Question 7.

Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0,\ 0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation. Then use the Cauchy-Riemann equations in polar form, to derive the harmonic conjugate $v(r, \theta) = \theta$.

SOLUTION: If $u(r, \theta) = \ln r$, for $r > 0,\ 0 < \theta < 2\pi$, then
$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}(\ln r)\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2},$$
and
$$\frac{\partial u}{\partial r} = \frac{1}{r}, \qquad \text{and} \qquad \frac{\partial^2 u}{\partial \theta^2} = 0.$$
Therefore,
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \left(-\frac{1}{r^2}\right) + \frac{1}{r}\left(\frac{1}{r}\right) + 0 = 0$$
for all $r > 0,\ 0 < \theta < 2\pi$, and u is harmonic in this region.

In order to find a harmonic conjugate $v(r, \theta)$ of u, we use the Cauchy-Riemann equations
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

and we want

$$\frac{\partial v}{\partial \theta} = r\left(\frac{1}{r}\right) = 1$$
$$\frac{\partial v}{\partial r} = 0$$

Integrating the first equation with respect to θ holding r fixed, we have

$$v(r,\theta) = \theta + h(r)$$

where $h(r)$ is an arbitrary function of r. Using the second equation, we have

$$\frac{\partial v}{\partial r} = h'(r) = 0,$$

that is, $h'(r) = 0$ for all $r > 0$, so that $h(r) = C$ (constant) for all $r > 0$.

Therefore, for any real constant C, the function

$$v(r,\theta) = \theta + C$$

is a harmonic conjugate for $u(r,\theta) = \ln r$ for $r > 0,\ 0 < \theta < 2\pi$.

So we may take $v(r,\theta) = \theta,\ r > 0,\ 0 < \theta < 2\pi$, as the harmonic conjugate of $u = \ln r$.

The function $f(z) = \ln r + i\,\theta,\ r > 0,\ 0 < \theta < 2\pi$, is analytic in this domain, and

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right) = e^{-i\theta}\cdot\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

for $r > 0,\ 0 < \theta < 2\pi$.

Question 8.

Show that (a) $\exp(2 \pm 3\pi\, i) = -e^2$; (b) $\exp\left(\dfrac{2+\pi\, i}{4}\right) = \sqrt{\dfrac{e}{2}}(1+i)$; (c) $\exp(z + \pi\, i) = -\exp z$.

SOLUTION:

(a) From the definition of the exponential function, we have

$$\exp(2 \pm 3\pi\, i) = e^2(\cos 3\pi \pm i\,\sin 3\pi) = e^2(-1 + i\,0) = -e^2.$$

(b) From the definition of the exponential function, we have

$$\exp\left(\frac{2+\pi\, i}{4}\right) = e^{\frac{1}{2}}\cdot e^{\pi\, i/4} = \sqrt{e}\,(\cos\pi/4 + i\,\sin\pi/4) = \sqrt{\frac{e}{2}}(1+i).$$

(c) From the definition of the exponential function, we have

$$\exp(z+\pi\, i) = e^{x+i(y+\pi)} = e^x\left(\cos(y+\pi) + i\,\sin(y+\pi)\right) = -e^x(\cos y + i\,\sin y) = -e^{x+iy} = -e^z.$$

Question 9. Use the Cauchy-Riemann equations to show that the function

$$f(z) = \exp\overline{z}$$

is not analytic anywhere.

SOLUTION: If $f(z) = \exp \overline{z} = e^x(\cos y - i\,\sin y)$, then $u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$, and

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y, \quad \frac{\partial v}{\partial y} = -e^x \cos y \\ \frac{\partial u}{\partial y} &= -e^x \sin y, \quad \frac{\partial v}{\partial x} = -e^x \sin y. \end{aligned}$$

The Cauchy-Riemann equations hold if and only if

$$\begin{aligned} 2e^x \cos y &= 0 \\ 2e^x \sin y &= 0, \end{aligned}$$

that is, if and only if $\sin y = \cos y = 0$. However, this is impossible, since $\sin^2 y + \cos^2 y = 1$. Therefore, there are **no** points $z \in \mathbb{C}$ for which $f(z) = e^{\overline{z}}$ is differentiable, and so **no** points $z \in \mathbb{C}$ at which f is analytic.

Question 10.

Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

SOLUTION: We have

$$|\exp(2z + i)| = |e^{2x+i(2y+1)}| = e^{2x}|e^{i(2y+1)}| = e^{2x},$$

and

$$|\exp(iz^2)| = |e^{i(x^2-y^2+2ixy)}| = e^{-2xy}|e^{i(x^2-y^2)}| = e^{-2xy},$$

therefore

$$|e^{2z+i} + e^{iz^2}| \le |e^{2z+i}| + |e^{iz^2}| = e^{2x} + e^{-2xy}.$$

Question 11.

Show that $|\exp(z^2)| \le \exp(|z|^2)$.

SOLUTION: We have

$$\left|e^{z^2}\right| = \left|e^{x^2-y^2} \cdot e^{2ixy}\right| = \left|e^{x^2-y^2}\right| \cdot \left|e^{2ixy}\right| = e^{x^2-y^2},$$

and since $x^2 - y^2 \le x^2 + y^2$, and the (real) exponential function is increasing, then $e^{x^2-y^2} \le e^{x^2+y^2}$, so that

$$\left|e^{z^2}\right| = e^{x^2-y^2} \le e^{x^2+y^2} = e^{|z|^2},$$

that is,

$$\left|e^{z^2}\right| \le e^{|z|^2}$$

for all $z \in \mathbb{C}$.

Question 12.

Find all values of z such that

(a) $e^z = -2$; (b) $e^z = 1 + \sqrt{3}\,i$; (c) $\exp(2z - 1) = 1$.

Ans:

(a) $z = \ln 2 + (2n + 1)\pi\, i \quad (n = 0, \pm 1, \pm 2, \dots)$.

(b) $z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi\, i \quad (n = 0, \pm 1, \pm 2, \dots)$.

(c) $z = \frac{1}{2} + n\pi\, i \quad (n = 0, \pm 1, \pm 2, \dots)$.

SOLUTION:

(a) Note that

$$e^z = -2 \quad \text{if and only if} \quad e^x \cdot e^{iy} = 2 \cdot e^{i\pi} \quad \text{if and only if} \quad e^x = 2 \quad \text{and} \quad e^{iy} = e^{i\pi}.$$

This last condition is true if and only if

$$x = \ln 2 \qquad \text{and} \qquad y = \pi + 2k\pi$$

for $k = 0, \pm 1, \pm 2, \ldots$, that is, if and only if

$$z = \ln 2 + (2k+1)\pi i$$

for $k = 0, \pm 1, \pm 2, \ldots$.

(b) Note that

$$e^z = 1 + \sqrt{3}\, i \quad \text{if and only if} \quad e^z = 2\left(\frac{1}{2} + \frac{\sqrt{3}\, i}{2}\right) \quad \text{if and only if} \quad e^z = 2 \cdot e^{i\pi/3}.$$

This last condition is true if and only if

$$x = \ln 2 \qquad \text{and} \qquad y = \frac{\pi}{3} + 2\pi k$$

for $k = 0, \pm 1, \pm 2, \ldots$, that is, if and only if

$$z = \ln 2 + i\left(\frac{\pi}{3} + 2\pi k\right)$$

for $k = 0, \pm 1, \pm 2, \ldots$.

(c) Note that

$$e^{(2z-1)} = 1 \quad \text{if and only if} \quad e^{2x-1} \cdot e^{2i\,y} = 1 \cdot e^{i\,0}$$

and this last condition is true if and only if

$$e^{2x-1} = 1 \qquad \text{and} \qquad 2y = 2\pi k,$$

for $k = 0, \pm 1, \pm 2, \ldots$, that is, if and only if

$$2x - 1 = \ln 1 = 0 \qquad \text{and} \qquad y = \pi k,$$

for $k = 0, \pm 1, \pm 2, \ldots$, that is, if and only if

$$z = \frac{1}{2} + \pi k i$$

for $k = 0, \pm 1, \pm 2, \ldots$.

Question 13. We showed in class that for the inversion mapping $f(z) = 1/z$, $z \neq 0$, the real and imaginary parts of $f(z)$ are

$$u(x, y) = \frac{x}{x^2 + y^2} \qquad \text{and} \qquad v(x, y) = \frac{-y}{x^2 + y^2}.$$

Show that the level curves of $u(x, y)$ are a family of circles passing through the origin with center on the real axis; while the level curves of $v(x, y)$ are a family of circles passing through the origin with center on the imaginary axis.

SOLUTION: If $u(x,y)$ is constant, say,

$$u(x,y) = \frac{x}{x^2+y^2} = \frac{1}{2k},$$

then

$$x^2+y^2 = 2kx, \qquad \text{that is,} \qquad (x-k)^2+y^2 = k^2.$$

Thus, the level curves of the real part of $f(z) = 1/z$ are a family of circles centered on the real axis and passing through the origin.

Similarly, If $v(x,y)$ is constant, say,

$$v(x,y) = \frac{-y}{x^2+y^2} = \frac{1}{2k},$$

then

$$x^2+y^2 = -2ky, \qquad \text{that is,} \qquad x^2+(y+k)^2 = k^2.$$

Thus, the level curves of the imaginary part of $f(z) = 1/z$ are a family of circles centered on the imaginary axis and passing through the origin.

Note that for any $z = x+iy \neq 0$, the gradients

$$\nabla u(x_0,y_0) = \left(\frac{\partial u}{\partial x}(x_0,y_0), \frac{\partial u}{\partial x}(x_0,y_0)\right)$$

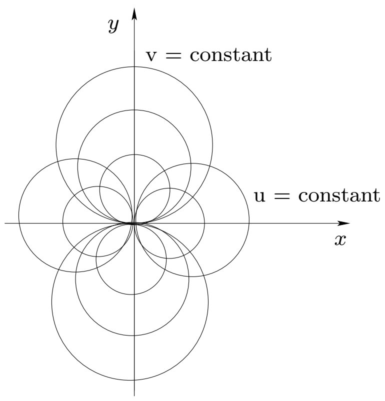
and

$$\nabla v(x_0,y_0) = \left(\frac{\partial v}{\partial x}(x_0,y_0), \frac{\partial v}{\partial x}(x_0,y_0)\right)$$

are perpendicular to the level curves of u and v, respectively, passing through the point (x_0,y_0). In fact, from the Cauchy-Riemann equations, the inner product

$$\frac{\partial u}{\partial x}(x_0,y_0) \cdot \frac{\partial v}{\partial x}(x_0,y_0) = \left.\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} + \left.\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} = 0,$$

Thus, the level curves for $u(x,y)$ and $v(x,y)$ intersect orthogonally at (x_0,y_0) as in the figure.



This is true in general for a function and its harmonic conjugate.