Math 309 - Spring-Summer 2017

Solutions to Problem Set # 3

Completion Date: Friday May 26, 2017

Question 1.

Show that $\lim_{z\to\infty} \dfrac{4z^2}{(z-1)^2} = 4.$

SOLUTION: We have

$$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = \lim_{z\to 0} \frac{4\,(1/z)^2}{(1/z-1)^2} = \lim_{z\to 0} \frac{4}{(1-z)^2} = 4.$$

Question 2.

Show that a set is unbounded if and only if every neighborhood of the point at infinity contains at least one point in S.

SOLUTION: If $S \subseteq \mathbb{C}$ is unbounded, then for each $n \geq 1$, there is a point $z_n \in S$ with $|z_n| \geq n$.

Now, given $\epsilon > 0$, choose $n_0 \geq 1$ with $0 < \dfrac{1}{n_0} < \epsilon$, then

$$|z_{n_0}| \geq n_0 > \frac{1}{\epsilon},$$

and z_{n_0} is in the ϵ-neighborhood $\{z : |z| > 1/\epsilon\}$ of the point of infinity. Thus, every neighborhood of the point at infinity contains at least one point in S.

Conversely, if every neighborhood of the point at infinity contains at least one point in S, then for each $n \geq 1$, we can choose a point $z_n \in S$ with $|z_n| \geq n$ (that is, z_n is in the $1/n$-neighborhood of the point at infinity). Then S cannot be bounded, since $z_n \in S$ for all $n \geq 1$, and $\lim_{n\to\infty} |z_n| = +\infty$, and there is **no** $M > 0$ such that $|z| \leq M$ for all $z \in S$.

Question 3.

Find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4;$ (b) $f(z) = \left(1 - 4z^2\right)^3;$

(c) $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq -1/2);$ (d) $f(z) = \dfrac{\left(1+z^2\right)^4}{z^2}$ $(z \neq 0).$

SOLUTION:

(a) $f'(z) = 6z - 2.$

(b) $f'(z) = 3(1-4z^2)^2(-8z) = -24z(1-4z^2)^2.$

(c) $f'(z) = \dfrac{1 \cdot (2z+1) - (z-1) \cdot 2}{(2z+1)^2} = \dfrac{3}{(2z+1)^2}$, for $z \neq -1/2$.

(d) $f'(z) = \dfrac{4(1+z^2)^3 \cdot 2z \cdot z^2 - (1+z^2)^4 \cdot 2z}{z^4} = \dfrac{2(1+z^2)^3}{z^3} \cdot (3z^2 - 1)$, for $z \neq 0$.

Question 4.

Apply the definition of the derivative to give a direct proof that

$$f'(z) = -\frac{1}{z^2} \qquad \text{when} \qquad f(z) = \frac{1}{z} \quad (z \neq 0).$$

Solution: If $f(z) = \dfrac{1}{z}$ for $z \neq 0$, then

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{z+h} - \dfrac{1}{z}}{h} = \lim_{h \to 0} \frac{z - (z+h)}{hz(z+h)},$$

that is,

$$f'(z) = \lim_{h \to 0} \frac{-h}{hz(z+h)} = \lim_{h \to 0} \frac{-1}{z(z+h)} = -\frac{1}{z^2}$$

for $z \neq 0$.

Question 5.

Show that $f'(z)$ does not exist at any point z when $f(z) = \operatorname{Im} z$.

Solution: Let $f(z) = \operatorname{Im}(z)$, then for $z, h \in \mathbb{C}$, with $h \neq 0$, we have

$$\frac{\operatorname{Im}(z+h) - \operatorname{Im}(z)}{h} = \frac{\operatorname{Im}(z) + \operatorname{Im}(h) - \operatorname{Im}(z)}{h} = \frac{\operatorname{Im}(h)}{h}.$$

Now, if $h \to 0$ through real values, $\operatorname{Im}(h) = 0$, and

$$\lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{\operatorname{Im}(z+h) - \operatorname{Im}(z)}{h} = \lim_{\substack{h \to 0 \\ h \text{ real}}} \frac{0}{h} = 0. \tag{1}$$

However, if $h \to 0$ through imaginary values, say $h = it$ where $t \in \mathbb{R}$ and $t \to 0$, then

$$\frac{\operatorname{Im}(h)}{h} = \frac{t}{it} = -i,$$

and

$$\lim_{\substack{h \to 0 \\ h \text{ imag}}} \frac{\operatorname{Im}(z+h) - \operatorname{Im}(z)}{h} = \lim_{\substack{h \to 0 \\ h \text{ imag}}} (-i) = -i. \tag{2}$$

Therefore, (1) and (2) imply that

$$\lim_{h \to 0} \frac{\operatorname{Im}(z+h) - \operatorname{Im}(z)}{h}$$

doesn't exist, that is, $f'(z)$ does not exist for any $z \in \mathbb{C}$.

Question 6.

Show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$; (b) $f(z) = z - \overline{z}$;

(c) $f(z) = 2x + i\,xy^2$; (d) $f(z) = e^x e^{-i\,y}$.

SOLUTION:

(a) If $f(z) = \overline{z} = x - iy$, then $u(x, y) = x$ and $v(x, y) = -y$, so that

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y},$$

and the Cauchy-Riemann equations do not hold at **any** point $z \in \mathbb{C}$. Therefore $f'(z)$ does not exist for any $z \in \mathbb{C}$.

(b) If $f(z) = z - \overline{z} = 2iy$, then $u(x, y) = 0$ and $v(x, y) = 2y$, so that

$$\frac{\partial u}{\partial x} = 0 \neq 2 = \frac{\partial v}{\partial y},$$

and again the Cauchy-Riemann equations do not hold at **any** point $z \in \mathbb{C}$. Therefore $f'(z)$ does not exist for any $z \in \mathbb{C}$.

(c) If $f(z) = 2x + ixy^2$, then $u(x, y) = 2x$ and $v(x, y) = xy^2$, so that

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2, & \frac{\partial v}{\partial y} &= 2xy \\ \frac{\partial u}{\partial y} &= 0, & \frac{\partial v}{\partial x} &= y^2. \end{aligned}$$

Now the Cauchy-Riemann equations are

$$\begin{aligned} 2 &= 2xy \\ 0 &= -y^2 \end{aligned}$$

and these equations have **no** solutions (x, y). Therefore, $f'(z)$ does not exist at any point $z \in \mathbb{C}$.

(d) If $f(z) = e^x e^{-i\,y} = e^x \cos y - ie^x \sin y$, then

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y, & \frac{\partial v}{\partial y} &= -e^x \cos y \\ \frac{\partial u}{\partial y} &= -e^x \sin y, & \frac{\partial v}{\partial x} &= -e^x \sin y. \end{aligned}$$

Now the Cauchy-Riemann equations are

$$\begin{aligned} 2e^x \cos y &= 0 \\ 2e^x \sin y &= 0 \end{aligned}$$

and since $e^x \neq 0$ for all $x \in \mathbb{R}$, these equations are

$$\cos y = 0 \qquad \text{and} \qquad \sin y = 0,$$

but this is impossible since $\cos^2 y + \sin^2 y = 1$. Therefore, there are **no** solutions to the Cauchy-Riemann equations, and $f'(z)$ does not exist for **any** $z \in \mathbb{C}$.

Question 7.

Determine where $f'(z)$ exists and find its value when

(a) $f(z) = \dfrac{1}{z}$; (b) $f(z) = x^2 + i\,y^2$; (c) $f(z) = z\,\mathrm{Im}\,z$;

Ans: (a) $f'(z) = -\dfrac{1}{z^2}$ $(z \neq 0)$; (b) $f'(x + i\,x) = 2x$; (c) $f'(0) = 0$.

SOLUTION:

(a) If $f(z) = \dfrac{1}{z}$, then

$$f(z) = \frac{\overline{z}}{|z|^2} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2},$$

so that

$$u(x,y) = \frac{x}{x^2+y^2} \qquad \text{and} \qquad v(x,y) = -\frac{y}{x^2+y^2}$$

for $z \neq 0$. Now,

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}.$$

Since the partial derivatives are all continuous at each $z \in \mathbb{C},\ z \neq 0$, and the Cauchy-Riemann equations hold at each $z \in \mathbb{C},\ z \neq 0$, then $f'(z)$ exists for all $z \neq 0$, and

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2ixy}{(x^2+y^2)^2},$$

that is,

$$f'(z) = -\frac{x^2-y^2-2ixy}{(x^2+y^2)^2} = -\frac{(\overline{z})^2}{|z|^4} = -\frac{1}{z^2}$$

for $z \neq 0$.

(b) If $f(z) = x^2 + i\,y^2$, then $u(x,y) = x^2$ and $v(x,y) = y^2$, and

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial y} = 2y$$
$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0,$$

so the Cauchy-Riemann equations hold only for those points $z = x + i\,y$ with $x = y$. Since all partial derivatives are continuous everywhere, $f'(z)$ exists only for the points $z = x + i\,x = x(1+i),\ x \in \mathbb{R}$, and

$$f'(x+ix) = \frac{\partial u}{\partial x}(x,x) + i\,\frac{\partial v}{\partial x}(x,x) = 2x.$$

(c) If $f(z) = z\mathrm{Im}(z) = (x + i\,y)\cdot y = xy + i\,y^2$, then $u(x,y) = xy$ and $v(x,y) = y^2$, so that

$$\frac{\partial u}{\partial x} = y, \qquad \frac{\partial v}{\partial y} = 2y$$
$$\frac{\partial u}{\partial y} = x, \qquad \frac{\partial v}{\partial x} = 0,$$

and the Cauchy-Riemann equations hold if and only if

$$y = 2y \qquad \text{and} \qquad x = 0,$$

that is, if and only if $x = y = 0$. Since all partial derivatives are continuous, then $f'(z)$ exists only for $z = 0$, and

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + i\,\frac{\partial v}{\partial x}(0,0) = 0.$$

Question 8.

Show that the function

$$f(z) = \sqrt{r}e^{i\,\theta/2} \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

is differentiable in the indicated domain of definition, and then find $f'(z)$.

Ans: $f'(z) = \dfrac{1}{2f(z)}$.

SOLUTION: If $f(z) = \sqrt{r}e^{i\theta/2}$, $r > 0,\ \alpha < \theta < \alpha + 2\pi$, then

$$f(z) = \sqrt{r}\,(\cos\theta/2 + i\,\sin\theta/2)\,,$$

so that $u(r,\theta) = \sqrt{r}\cos\theta/2$ and $v(r,\theta) = \sqrt{r}\sin\theta/2$.

Now,

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\theta/2 \qquad \text{and} \qquad \frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2}\cos\theta/2,$$

so that

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\theta/2 = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

for $r > 0,\ \alpha < \theta < \alpha + 2\pi$.

Also,

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2}\sin\theta/2 \qquad \text{and} \qquad \frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\theta/2,$$

so that

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{2\sqrt{r}}\sin\theta/2 = -\frac{\partial v}{\partial r}$$

for $r > 0,\ \alpha < \theta < \alpha + 2\pi$.

The partial derivatives are all continuous for each (r,θ) with $r > 0$, $\alpha < \theta < \alpha + 2\pi$, and the Cauchy-Riemann equations hold for each such point, so that $f'(z)$ exists for all such points, and

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right) = \frac{1}{2\sqrt{r}}\,(\cos\theta/2 + i\,\sin\theta/2)\,e^{-i\theta} = \frac{1}{2\sqrt{r}}e^{-i\theta/2} = \frac{1}{2f(z)},$$

for $r > 0,\ \alpha < \theta < \alpha + 2\pi$.

Question 9.

Show that when $f(z) = x^3 + i\,(1-y)^3$, it is legitimate to write

$$f'(z) = u_x + i\,v_x = 3x^2$$

only when $z = i$.

SOLUTION: If $f(z) = x^3 + i(1-y)^3$, then $u(x,y) = x^3$ and $v(x,y) = (1-y)^3$, so that

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2, & \frac{\partial v}{\partial y} &= -3(1-y)^2 \\ \frac{\partial u}{\partial y} &= \quad 0, & \frac{\partial v}{\partial x} &= 0. \end{aligned}$$

The Cauchy-Riemann equations become

$$3x^2 + 3(1-y)^2 = 0$$

and these hold **only** at the point $z = (0,1) = i$. Since all partial derivatives are continuous everywhere, then $f'(i)$ exists, and

$$f'(i) = \frac{\partial u}{\partial x}(0,1) + i\,\frac{\partial v}{\partial x}(0,1) = 0.$$

Question 10.

(a) Recall that if $z = x + i\,y$ then

$$x = \frac{z+\overline{z}}{2} \qquad \text{and } y = \frac{z-\overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x,y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y}\right).$$

(b) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary parts of a function $f(z) = u(x,y) + i\,v(x,y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[(u_x - v_y) + i\,(v_x + u_y)\right] = 0.$$

Thus derive the *complex form* $\dfrac{\partial f}{\partial \overline{z}} = 0$ *of the Cauchy-Riemann equations.*

SOLUTION:

(a) Since $z = x + i\,y$ and $\overline{z} = x - i\,y$, then

$$x = \frac{z+\overline{z}}{2} \qquad \text{and} \qquad y = \frac{z-\overline{z}}{2i},$$

and

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\frac{\partial F}{\partial x} + \left(-\frac{1}{2i}\right)\cdot\frac{\partial F}{\partial y} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y}\right).$$

(b) Now, if $f(z) = u(x,y) + i\,v(x,y)$ and the real-valued functions u and v satisfy the Cauchy-Riemann equations, then from part (a), we have

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\frac{\partial}{\partial x}(u + i\,v) + \frac{i}{2}\frac{\partial}{\partial y}(u + i\,v) = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y},$$

that is,

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,$$

since $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.