Math 309 - Spring-Summer 2017

Solutions to Problem Set # 2

Completion Date: Friday May 19, 2017

Question 1.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

(a) $(-16)^{1/4}$; (b) $\left(-8-8\sqrt{3}\,i\right)^{1/4}$.

Ans: (a) $\pm\sqrt{2}\,(1+i)$, $\pm\sqrt{2}\,(1-i)$; (b) $\pm(\sqrt{3}-i)$, $\pm(1+\sqrt{3}\,i)$.

SOLUTION:

(a) Note that

$$-16 = 16e^{i[\pi+2k\pi]}$$

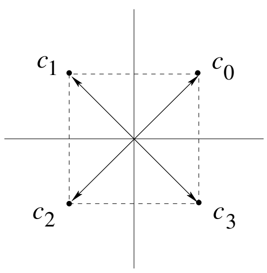
for $k = 0, \pm1, \pm2, \ldots$, so the four fourth roots of -16 are

$$c_k = 2e^{i[\pi/4+k\pi/2]}$$

for $k = 0, 1, 2, 3$. Therefore,

$$\begin{aligned}
c_0 &= 2e^{i\pi/4} = 2\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \sqrt{2}(1+i) \\
c_1 &= 2e^{i(\pi/4+\pi/2)} = 2e^{i3\pi/4} = 2\left[-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = -\sqrt{2}(1-i) \\
c_2 &= 2e^{i(\pi/4+\pi)} = 2e^{i5\pi/4} = -2\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = -\sqrt{2}(1+i) \\
c_3 &= 2e^{i(\pi/4+3\pi/2)} = 2e^{i7\pi/4} = -2\left[-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \sqrt{2}(1-i)
\end{aligned}$$

The four roots are the vertices of a square centered at the origin with side of length $2\sqrt{2}$ as shown in the figure, the principal root is $c_0 = \sqrt{2}(1+i)$.



(b) Note that

$$-8-8\sqrt{3}\,i=-16\left(\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)=16e^{i\pi}e^{i\pi/3}=16e^{i4\pi/3}=16e^{i[4\pi/3+2\pi k]}$$

for $k=0,\,\pm1,\,\pm2,\,\ldots$, so the four fourth roots of $-8-8\sqrt{3}\,i$ are

$$c_k=2e^{i[\pi/3+\pi k/2]}$$

for $k=0,1,2,3$. Therefore,

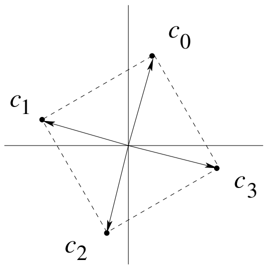
$$c_0=2e^{i\pi/3}=2\left(\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)=1+\sqrt{3}\,i$$

$$c_1=2e^{i[\pi/3+\pi/2]}=2e^{i5\pi/6}=2e^{i\pi}e^{-i\pi/6}=-2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)=-\sqrt{3}+i$$

$$c_2=2e^{i[\pi/3+\pi]}=2e^{i\pi}e^{i\pi/3}=-2\left(\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)=-(1+\sqrt{3}\,i)$$

$$c_3=2e^{i[\pi/3+3\pi/2]}=2e^{i11\pi/6}=2e^{i2\pi}e^{-i\pi/6}=2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)=\sqrt{3}-i$$

The four roots are the vertices of a square centered at the origin with side of length $2\sqrt{2}$ as shown in the figure, the principal root is $c_0=1+\sqrt{3}\,i$.



Note: These roots are probably not in the same order as the roots you found if you used the principal argument of $-8-8\sqrt{3}\,i$ as $-2\pi/3$, and the principal root would be $c_3=\sqrt{3}-i$ in this case.

Question 2.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; (b) $8^{1/6}$.

Ans: (b) $\pm\sqrt{2}$, $\pm\dfrac{1+\sqrt{3}\,i}{\sqrt{2}}$, $\pm\dfrac{1-\sqrt{3}\,i}{\sqrt{2}}$.

Solution:

(a) Note that

$$-1=e^{i\pi}=e^{i[\pi+2k\pi]}$$

for $k=0,\,\pm1,\,\pm2,\,\ldots$, so the three cube roots of -1 are

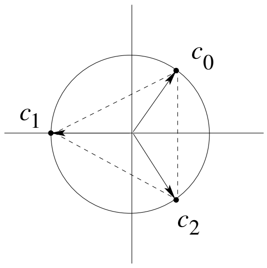
$$c_k=e^{i[\pi/3+2k\pi/3]}$$

for $k=0,1,2$.

Therefore,

$$
\begin{aligned}
c_0 &= e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}\,i}{2} \\
c_1 &= e^{i[\pi/3+2\pi/3]} = e^{i\pi} = -1 \\
c_2 &= e^{i[\pi/3+4\pi/3]} = e^{i5\pi/3} = e^{i2\pi}e^{-i\pi/3} = \frac{1}{2} - \frac{\sqrt{3}\,i}{2}
\end{aligned}
$$

The three third roots of -1 are the vertices of an equilateral triangle inscribed in the unit circle, as shown in the figure below, the principal root is $c_0 = \frac{1}{2} + \frac{\sqrt{3}\,i}{2}$.



(b) Note that

$$8 = 8e^{i0} = 8e^{i2k\pi}$$

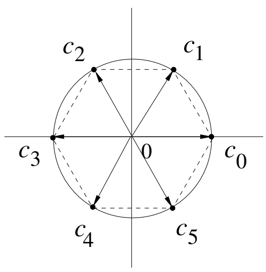
for $k = 0, \pm 1, \pm 2, \dots$, so the six sixth roots of 8 are

$$c_k = \sqrt{2}e^{i2k\pi/6} = \sqrt{2}e^{ik\pi/3}$$

for $k = 0, 1, 2, 3, 4, 5$. Therefore,

$$
\begin{aligned}
c_0 &= \sqrt{2}e^{0} = \sqrt{2} \\
c_1 &= \sqrt{2}e^{i\pi/3} = \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}\,i}{2}\right) = \frac{1+\sqrt{3}\,i}{\sqrt{2}} \\
c_2 &= \sqrt{2}e^{i2\pi/3} = \sqrt{2}e^{i\pi}e^{-i\pi/3} = -\sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right) = \frac{-1+\sqrt{3}\,i}{\sqrt{2}} \\
c_3 &= \sqrt{2}e^{i3\pi/3} = \sqrt{2}e^{i\pi} = -\sqrt{2} \\
c_4 &= \sqrt{2}e^{i4\pi/3} = \sqrt{2}e^{i\pi}e^{i\pi/3} = -\sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}\,i}{2}\right) = -\frac{1+\sqrt{3}\,i}{\sqrt{2}} \\
c_5 &= \sqrt{2}e^{i5\pi/3} = \sqrt{2}e^{i2\pi}e^{-i\pi/3} = \sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right) = \frac{1-\sqrt{3}\,i}{\sqrt{2}}
\end{aligned}
$$

The six sixth roots of 8 are the vertices of a regular hexagon inscribed in a circle of radius $\sqrt{2}$, and are shown in the figure below.



The principal root is $c_0 = \sqrt{2}$.

Question 3.

Sketch the following sets and determine which are domains:

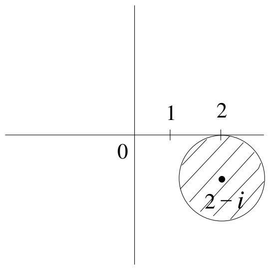
(a) $|z-2+i| \leq 1$; (b) $|2z+3| > 4$; (c) $\text{Im } z > 1$;

(d) $\text{Im } z = 1$; (e) $0 \leq \arg z \leq \pi/4$ $(z \neq 0)$; (f) $|z-4| \geq |z|$.

Ans: (b), (c) are domains.

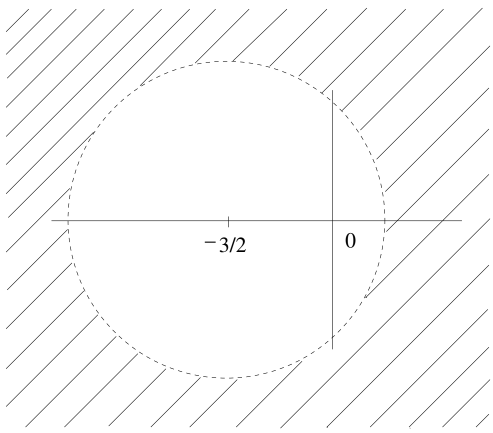
SOLUTION:

(a) The set $A = \{z \in \mathbb{C} : |z-2+i| \leq 1\}$ is the closed disk of radius 1 centered at the point $z_0 = 2 - i$, and is *not* a domain.



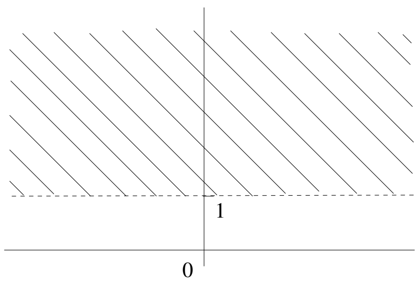
It is connected, but is not open, since for example, the point $z = 2$ is in A, but is not an interior point of A. (For any $\epsilon > 0$, the ϵ-neighborhood of $z = 2$ contains points that are not in A)

(b) The set $B = \{z \in \mathbb{C} : |2z+3| > 4\}$ is the exterior of the closed disk of radius 2 centered at the point $z_0 = -\dfrac{3}{2}$, and it *is* a domain.



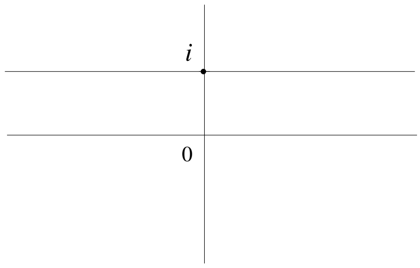
It is open and connected, and is therefore a domain.

(c) The set $C = \{z \in \mathbb{C} : \text{Im } z > 1\}$ is the half-plane $y > 1$, and it *is* a domain.



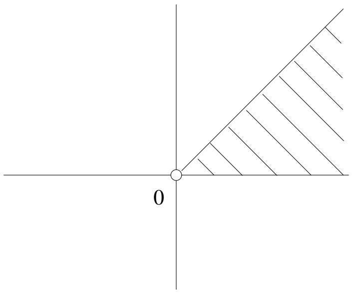
It is open and connected, and is therefore a domain.

(d) The set $D = \{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ is the set of points $z = x + iy$ where $y = 1$, and it is *not* a domain.



It is connected, but it is not open, since for example, the point $z = i$ is not an interior point of D. (For any $\epsilon > 0$, the ϵ-neighborhood of $z = i$ contains points that are not in D)

(e) The set $E = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/4\}$ is the set of all nonzero points in the first quadrant between the real axis and the line $y = x$, and it is *not* a domain.



It is connected, but it is not open, since for example, any nonzero point on the real axis is not an interior point of E.

(f) The set $F = \{z \in \mathbb{C} : |z - 4| \geq |z|\}$ is the set of all points z such that the distance from z to 4 is greater than or equal to the distance from z to 0, and this is precisely the set of points $z = x + iy$ such that $x \leq 2$, that is, the half-plane $x \leq 2$.

To see this, note that since the absolute value is a nonnegative real number, then

$$|z - 4| \geq |z| \quad \text{if and only if} \quad |z - 4|^2 \geq |z|^2,$$

that is, if and only if

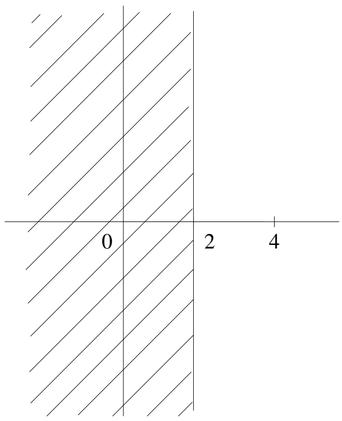
$$(x - 4)^2 + y^2 \geq x^2 + y^2,$$

that is, if and only if

$$-8x + 16 \geq 0,$$

that is, if and only if

$$x \leq 2.$$



Again, the set F is connected but is not open, so that F is *not* a domain.

Question 4.

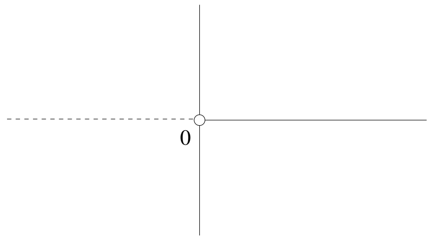
In each case, sketch the closure of the set:

(a) $-\pi < \arg z < \pi$ $(z \neq 0)$; (b) $|\text{Re } z| < |z|$;

(c) $\text{Re}\left(\dfrac{1}{z}\right) \leq \dfrac{1}{2}$; (d) $\text{Re}\left(z^2\right) > 0$.

SOLUTION:

(a) The set $A = \{z \in \mathbb{C} \,:\, -\pi < \arg z < \pi,\ z \neq 0\}$ consists of the entire complex plane **except** for the negative real axis and the point 0.



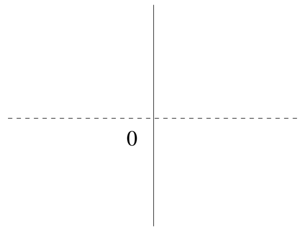
The closure of A is the entire complex plane since the boundary of A is just

$$\text{bdy}(A) = \{z \in \mathbb{C} \,:\, z = x,\ x \leq 0\}$$

and $\text{cl}(A) = A \cup \text{bdy}(A) = \mathbb{C}$.

(b) The set $B = \{z \in \mathbb{C} \,:\, |\text{Re } z| < |z|\}$ consists of the entire complex plane **except** the real axis $y = 0$, since

$$|x| < \sqrt{x^2 + y^2} \quad \text{if and only if} \quad x^2 < x^2 + y^2 \quad \text{if and only if} \quad y^2 > 0 \quad \text{if and only if} \quad y \neq 0.$$



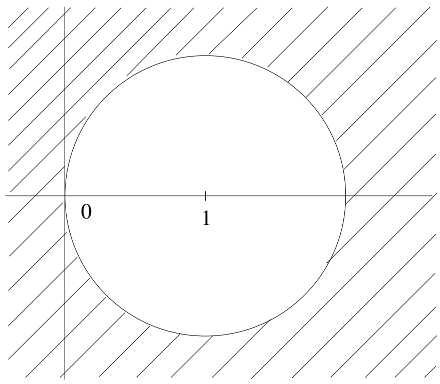
The closure of B is the entire complex plane since the boundary of B is just the real axis

$$\text{bdy}(B) = \{z \in \mathbb{C} \,:\, z = x,\ -\infty < x < \infty\}$$

and $\text{cl}(B) = B \cup \text{bdy}(B) = \mathbb{C}$.

(c) The set $C = \left\{z \in \mathbb{C} \,:\, \text{Re}\left(\dfrac{1}{z}\right) \leq \dfrac{1}{2}\right\}$ consists of the exterior of the open disk centered at $z = 1$ with radius 1, since

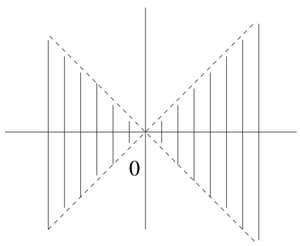
$$\text{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2 + y^2} \leq \frac{1}{2} \quad \text{if and only if} \quad (x-1)^2 + y^2 \geq 1.$$



Since the open disk is an open set, then its complement is closed, and therefore $\mathrm{cl}(C) = C$.

(d) The set $D = \{z \in \mathbb{C} : \mathrm{Re}(z^2) > 0\}$ consists of the points lying strictly between the line $y = x$ and the line $y = -x$, not including the origin, since

$$\mathrm{Re}(x^2 - y^2 + 2ixy) > 0 \quad \text{if and only if} \quad x^2 - y^2 > 0 \quad \text{if and only if} \quad |x| > |y|.$$



The closure is the entire wedge-shaped region since

$$\mathrm{bdy}(D) = \{z \in \mathbb{C} : z = x(1+i), \ -\infty < x < \infty\} \cup \{z \in \mathbb{C} : z = x(1-i), \ -\infty < x < \infty\},$$

and $\mathrm{cl}(D) = \{z \in \mathbb{C} : \mathrm{Re}(z^2) \geq 0\} = \{z \in \mathbb{C} : z = x + iy, \ |x| \geq |y|\}$.

Question 5.

Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x,y) + i\,v(x,y)$.

Ans: $(x^3 - 3xy^2 + x + 1) + i\,(3x^2y - y^3 + y)$.

Solution: If $z = x + iy$, then

$$f(z) = (x+iy)^3 + (x+iy) + 1 = (x+iy)(x^2 - y^2 + 2ixy) + x + iy + 1,$$

that is,

$$f(z) = x^3 - xy^2 + 2ix^2y + ix^2y - iy^3 - 2xy^2 + x + iy + 1,$$

that is,

$$f(z) = x^3 - 3xy^2 + x + 1 + i(3x^2y - y^3 + y).$$

Therefore, $f(z) = u(x,y) + iv(x,y)$, where

$$u(x,y) = x^3 - 3xy^2 + x + 1 \qquad \text{and} \qquad v(x,y) = 3x^2y - y^3 + y.$$

Question 6. Suppose that $f(z) = x^2 - y^2 - 2y + i\,(2x - 2xy)$, where $z = x + i\,y$. Use the expressions

$$x = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad y = \frac{z - \overline{z}}{2i}$$

to write $f(z)$ in terms of z and simplify the result.

Ans: $\overline{z}^2 + 2i\,z$.

SOLUTION: We have

$$\begin{aligned} f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\ &= x^2 - y^2 - 2ixy + i2x - 2y \\ &= (x - iy)^2 + i(2x + 2iy) \\ &= \overline{z}^2 + 2iz, \end{aligned}$$

so that $f(z) = \overline{z}^2 + 2iz$.

Question 7. Find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, and $v = 2$.

SOLUTION: Under the transformation $w = f(z) = z^2$, that is,

$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy, \end{aligned}$$

the vertical line $u = 1$ in the w-plane is the image of the right branch of the hyperbola $x^2 - y^2 = 1$, while the vertical line $u = 2$ in the w-plane is the image of the right branch of the hyperbola $x^2 - y^2 = 2$. Therefore, the vertical strip between $u = 1$ and $u = 2$ is the image under $w = z^2$ of the region between the two hyperbolae $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$.

The horizontal line $v = 1$ in the w-plane is the image of the upper branch of the hyperbola $2xy = 1$, while the horizontal line $v = 2$ in the w-plane is the image of the upper branch of the hyperbola $2xy = 2$. Therefore, the horizontal strip between $v = 1$ and $v = 2$ is the image under $w = z^2$ of the region between the two hyperbolae $2xy = 1$ and $2xy = 2$.

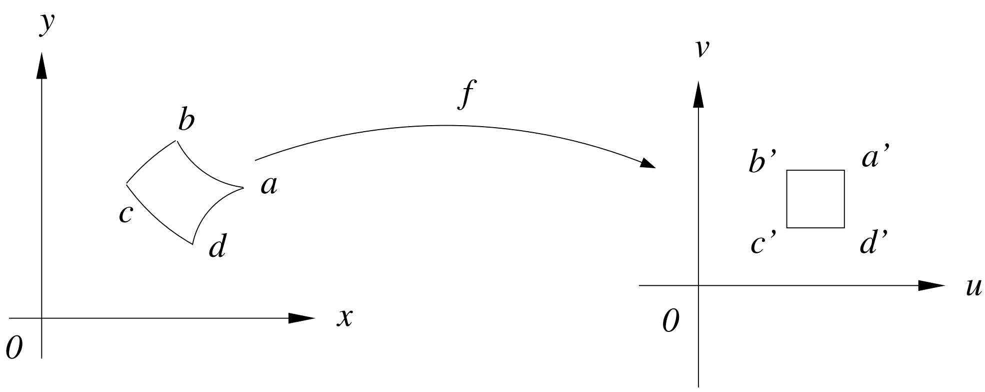
The domain

$$T = \{(u, v) \,:\, 1 < u < 2,\, 1 < v < 2\}$$

in the w-plane is the image under the map $w = z^2$ of the domain

$$S = \{(x, y) \,:\, 1 < x^2 - y^2 < 2\} \cap \{(x, y) \,:\, 1 < 2xy < 2\}$$

in the z-plane. The regions are sketched below.



Question 8. Sketch the region onto which the sector $r \le 1,\ 0 \le \theta \le \pi/4$ is mapped by the transformation

(a) $w = z^2$; (b) $w = z^3$; (c) $w = z^4$.

Solution: If $w = \rho e^{i\phi}$, and if $z = re^{i\theta}$, where $r \le 1,\ 0 \le \theta \le \pi/4$, then

(a) For $w = z^2$, we have $w = r^2 e^{i2\theta}$, so that $\rho = r^2$ and $\phi = 2\theta$, and

$$\rho = r^2 \le 1 \qquad \text{and} \qquad 0 \le \phi \le 2\pi/4 = \pi/2.$$

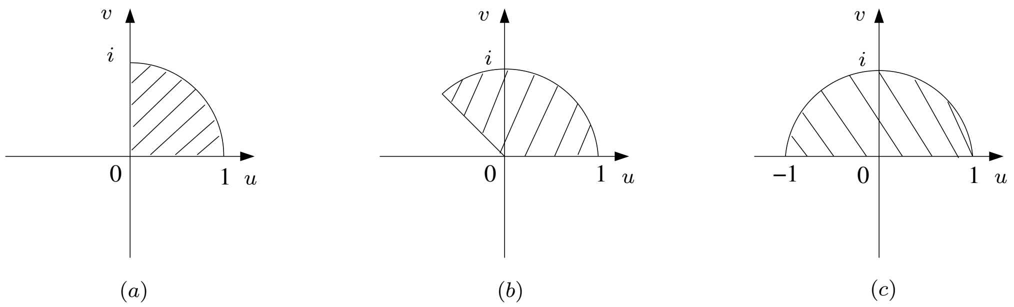
(b) For $w = z^3$, we have $w = r^3 e^{i3\theta}$, so that $\rho = r^3$ and $\phi = 3\theta$, and

$$\rho = r^3 \le 1 \qquad \text{and} \qquad 0 \le \phi \le 3\pi/4.$$

(c) For $w = z^4$, we have $w = r^4 e^{i4\theta}$, so that $\rho = r^4$ and $\phi = 4\theta$, and

$$\rho = r^4 \le 1 \qquad \text{and} \qquad 0 \le \phi \le 4\pi/4 = \pi.$$

The regions are as shown below.



Question 9.

(a) Describe and sketch the set

$$\mathcal{D} = \left\{ z \in \mathbb{C} \mid 2\,\mathrm{Re}(z^2) = |z|^2 \right\}.$$

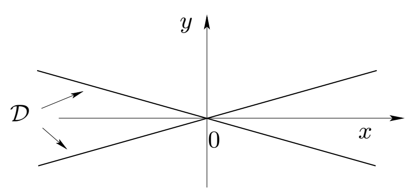
(b) Describe and sketch the set

$$\mathcal{D} = \left\{ z \in \mathbb{C} \mid \mathrm{Im}\left(\frac{1}{z}\right) > 1 \right\}.$$

Solution:

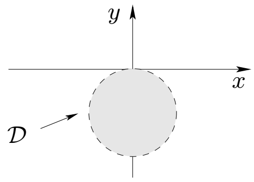
(a) Note that $2\,\mathrm{Re}(z^2) = |z|^2$ if and only if $2(x^2 - y^2) = x^2 + y^2$, if and only if $x^2 = 3y^2$, if and only if, $|y| = \frac{1}{\sqrt{3}}|x|$, if and only if $y = \pm\frac{1}{\sqrt{3}}x$.

Therefore, $z = x + iy$ is in $\mathcal{D}$ if and only if z is on one of the lines $y = \frac{1}{\sqrt{3}}x$ or $y = -\frac{1}{\sqrt{3}}x$, as in the figure below.



(b) Note that $\operatorname{Im}\left(\frac{1}{z}\right) > 1$ if and only if $-\dfrac{y}{x^2+y^2} > 1$, if and only if $-y > x^2 + y^2$, if and only if $x^2 + \left(y + \frac{1}{2}\right)^2 < \frac{1}{4}$.

Therefore, $z = x + iy$ is in $\mathcal{D}$ if and only if z is in the **interior** of the disk centered at $\left(0, -\frac{1}{2}\right)$ with radius $\frac{1}{2}$, as in the figure below.



Question 10.

(a) Given a positive integer $n > 2$, find all complex numbers $z \in \mathbb{C}$ satisfying

$$\overline{z} = z^{n-1}.$$

(b) Let ω_n be the primitive n^{th} root of unity given by $e^{\frac{2\pi i}{n}}$, $n \geq 2$, calculate

$$1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1}.$$

SOLUTION:

(a) If $z \neq 0$ and $n > 2$, then $\overline{z} = z^{n-1}$ if and only if $z\overline{z} = z^n$, if and only if $|z|^2 = z^n$. However, if $|z|^2 = z^n$, then $|z|^2 = |z^n| = |z|^n$, and since $n > 2$, this implies that $|z| = 1$, and therefore, $z^n = |z|^2 = 1$.

Conversely, if $z^n = 1$, then $|z^n| = |z|^n = 1$, so that $|z| = 1$, and hence $|z|^2 = z^n$.

Thus, the solutions to $\overline{z} = z^{n-1}$ are $z = 0$ **and** the n^{th} roots of unity $z_k = e^{\frac{2\pi i k}{n}}$, $k = 0, 1, 2, \ldots, n-1$.

(b) If $w_n = e^{\frac{2\pi i}{n}}$, then

$$\begin{aligned}(1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1})(1 - \omega_n) &= 1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1} \\ &\quad - \omega_n - 2\,\omega_n^2 - \cdots - (n-1)\,\omega_n^{n-1} - n\,\omega_n^n \\ &= 1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} - n \\ &= 0 - n = -n.\end{aligned}$$

Therefore,

$$1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1} = \frac{n}{\omega_n - 1},$$

since $\omega_n \neq 1$.