Math 309 - Spring-Summer 2017

Solutions to Problem Set # 12

Completion Date: Friday August 4, 2017

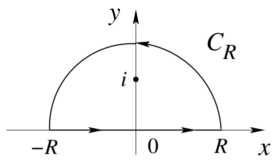
Question 1.

Use residues to evaluate the improper integral

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

Ans: $\pi/4$.

SOLUTION: Let $f(z) = \dfrac{1}{(1+z^2)^2}$, and for $R > 1$, consider the integral of f over the contour C_R shown below.



We have

$$\int_{C_R} \frac{dz}{(1+z^2)^2} + \int_{-R}^0 \frac{dx}{(1+x^2)^2} + \int_0^R \frac{dx}{(1+x^2)^2} = 2\pi i \operatorname*{Res}_{z=i}\left(\frac{1}{(1+z^2)^2}\right),$$

that is,

$$\int_{C_R} \frac{dz}{(1+z^2)^2} + 2\int_0^R \frac{dx}{(1+x^2)^2} = 2\pi i \operatorname*{Res}_{z=i}\left(\frac{1}{(1+z^2)^2}\right).$$

Now,

$$f(z) = \frac{1}{(1+z^2)^2} = \frac{\Phi(z)}{(z-i)^2}$$

where $\Phi(z) = \dfrac{1}{(z+i)^2}$ is analytic at $z = i$ and $\Phi(i) = -\dfrac{1}{4} \neq 0$, so that f has a pole of order $m = 2$ at $z = i$, with

$$\operatorname*{Res}_{z=i}(f(z)) = \Phi'(i) = \left.\frac{-2}{(z+i)^3}\right|_{z=i} = -\frac{2}{(2i)^3} = -\frac{i}{4}.$$

Therefore,

$$\int_{C_R} \frac{dz}{(1+z^2)^2} + 2\int_0^R \frac{dx}{(1+x^2)^2} = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2} \qquad (*)$$

However, on C_R, we have $z = Re^{i\theta}$, and

$$|1 + z^2|^2 \ge \left(|z|^2 - 1\right)^2 \ge (R^2 - 1)^2$$

if $R > 1$, so that

$$\left| \int_{C_R} \frac{dz}{(1+z^2)^2} \right| \le \frac{2\pi R}{(R^2-1)^2} \to 0$$

as $R \to \infty$.

Letting $R \to \infty$ in $(*)$, we have

$$2\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{2},$$

that is,

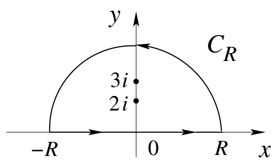
$$\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}.$$

Question 2. Use residues to evaluate the improper integral

$$\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}.$$

Ans: $\pi/200$.

Solution: Let $f(z) = \dfrac{z^2}{(z^2+9)(z^2+4)^2}$, and for $R > 3$, consider the integral of f over the contour C_R shown below.



We have

$$\int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2}\,dz + 2\int_0^R \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = 2\pi i \left[\operatorname*{Res}_{z=2i}(f(z)) + \operatorname*{Res}_{z=3i}(f(z))\right].$$

Now, $f(z)$ has a simple pole at $z = 3i$ with

$$\operatorname*{Res}_{z=3i}(f(z)) = \frac{(3i)^2}{(3i+3i)((3i)^2+4)^2} = -\frac{3}{50i},$$

and $f(z)$ has a pole of order 2 at $z = 2i$ with

$$\operatorname*{Res}_{z=2i}(f(z)) = \frac{d}{dz}\left[\frac{z^2}{(z^2+9)(z+2i)}\right]\Bigg|_{z=2i} = \frac{13}{200i}.$$

Therefore,

$$\int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2}\,dz + 2\int_0^R \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = 2\pi i\left[-\frac{12}{200i}+\frac{13}{200i}\right] = \frac{2\pi}{200},$$

that is,

$$\int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2}\,dz + 2\int_0^R \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = \frac{2\pi}{200}. \qquad (**)$$

On C_R, we have

$$f(z) \le \frac{R^2}{(R^2-9)(R^2-4)^2},$$

so that

$$\left|\int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2}\,dz\right| \le \frac{\pi R^3}{(R^2-9)(R^2-4)^2} \to 0$$

as $R\to\infty$.

Letting $R\to\infty$ in $(**)$, we get

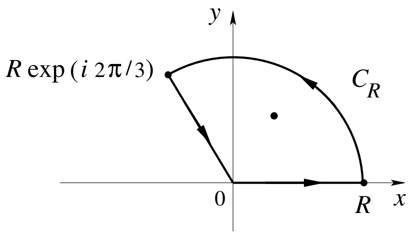
$$2\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = \frac{2\pi}{200},$$

that is,

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = \frac{\pi}{200}.$$

Question 3. Use residues and the contour shown in the figure below, where $R>1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$



Solution: Let $f(z) = \dfrac{1}{z^3+1}$, and consider the integral of f around the contour C_R with $R>1$, as shown above.

Since f is analytic inside and on this contour except for a simple pole at $z=e^{i\pi/3}$, then

$$\int_{C_R}\frac{dz}{z^3+1} = \int_0^R \frac{dx}{x^3+1} + \int_0^{2\pi/3}\frac{iRe^{i\theta}\,d\theta}{R^3e^{3i\theta}+1} + \int_R^0 \frac{e^{2\pi i/3}dr}{r^3e^{2\pi i}+1} = 2\pi i \operatorname*{Res}_{z=e^{\pi i/3}}\left(\frac{1}{z^3+1}\right),$$

that is,

$$\int_0^R \frac{dx}{x^3+1} + \int_0^{2\pi/3}\frac{iRe^{i\theta}\,d\theta}{R^3e^{3i\theta}+1} - \int_0^R \frac{e^{2\pi i/3}dr}{r^3+1} = 2\pi i \operatorname*{Res}_{z=e^{\pi i/3}}\left(\frac{1}{z^3+1}\right),$$

and therefore

$$\int_0^{2\pi/3}\frac{iRe^{i\theta}\,d\theta}{R^3e^{3i\theta}+1} + \left(1-e^{2\pi i/3}\right)\int_0^R \frac{dx}{x^3+1} = 2\pi i \operatorname*{Res}_{z=e^{\pi i/3}}\left(\frac{1}{z^3+1}\right).$$

Now, we have

$$\operatorname*{Res}_{z=e^{\pi i/3}}\left(\frac{1}{z^3+1}\right)=\frac{1}{3z^2}\bigg|_{z=e^{\pi i/3}}=\frac{e^{-2\pi i/3}}{3},$$

so that

$$\int_0^{2\pi/3}\frac{iRe^{i\theta}\,d\theta}{R^3e^{3i\theta}+1}+\left(1-e^{2\pi i/3}\right)\int_0^R\frac{dx}{x^3+1}=2\pi i\cdot\frac{e^{-2\pi i/3}}{3}. \qquad (***)$$

On the circular arc $z=Re^{i\theta}$, $0\le\theta\le 2\pi/3$, and we have

$$\left|\int_0^{2\pi i/3}\frac{iRe^{i\theta}\,d\theta}{R^3e^{3i\theta}+1}\right|\le\frac{2\pi R}{3}\cdot\frac{R}{R^3-1}\to 0$$

as $R\to\infty$, so that from $(***)$ we get

$$\left(1-e^{2\pi i/3}\right)\int_0^\infty\frac{dx}{x^3+1}=2\pi i\cdot\frac{e^{-2\pi i/3}}{3},$$

that is,

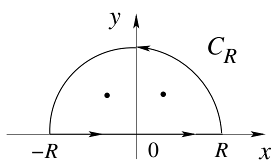
$$\int_0^\infty\frac{dx}{x^3+1}=\frac{2\pi i\cdot e^{-2\pi i/3}}{3\left(1-e^{2\pi i/3}\right)}=\frac{2\pi i(-1-\sqrt{3}\,i)}{3(3-\sqrt{3}\,i)}=\frac{2\pi}{3\sqrt{3}}.$$

Question 4. Use residues to evaluate the improper integral

$$\int_{-\infty}^\infty\frac{x\sin ax}{x^4+4}\,dx \quad (a>0).$$

Ans: $\dfrac{\pi}{2}e^{-a}\sin a$.

Solution: Let $f(z)=\dfrac{ze^{iaz}}{z^4+4}$, and consider the integral of f around the contour shown below, where $R>\sqrt{2}$.



Now, f is analytic inside and on the contour except at

$$z_1=\sqrt{2}e^{i\pi/4}=1+i \qquad\text{and}\qquad z_2=\sqrt{2}e^{3i\pi/4}=-1+i$$

and f has simple poles at these points with

$$\operatorname*{Res}_{z=z_1}(f(z))=\frac{z_1e^{iaz_1}}{4z_1^3}=\frac{e^{ia(1+i)}}{4(1+i)^2}=\frac{1}{8i}e^{-a}\cdot e^{ia},$$

and

$$\operatorname*{Res}_{z=z_2}(f(z))=\frac{z_2e^{iaz_2}}{4z_2^3}=\frac{e^{ia(-1+i)}}{4(-1+i)^2}=-\frac{1}{8i}e^{-a}\cdot e^{-ia}.$$

Therefore,

$$\int_{-R}^{R} \frac{xe^{iax}}{x^4+4}\,dx + \int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz = 2\pi i\left[\frac{e^{-a}}{8i}\left(e^{ia}-e^{-ia}\right)\right] = \frac{\pi i}{2}e^{-a}\sin a. \qquad (****)$$

Now, on C_R we have

$$e^{iaRe^{i\theta}} = e^{iaR(\cos\theta + i\sin\theta)} = e^{-aR\sin\theta}\cdot e^{iaR\cos\theta},$$

so that

$$\left|\frac{ze^{iaz}}{z^4+4}\right| \le \frac{Re^{-aR\sin\theta}}{R^4-4}$$

on C_R, and from Jordan's inequality, we have

$$\left|\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz\right| \le \frac{2R^2}{R^4-4}\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta < \frac{\pi R}{R^4-4} \to 0$$

as $R\to\infty$. Letting $R\to\infty$ in $(****)$, we have

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^4+4}\,dx = \frac{\pi i}{2}e^{-a}\sin a,$$

and therefore,

$$\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a.$$

Question 5. Use residues to find the Cauchy principal value of the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4x+5}.$$

Ans: $-\dfrac{\pi}{e}\sin 2$.

Solution: We write

$$f(z) = \frac{1}{z^2+4z+5} = \frac{1}{(z-z_1)(z-\overline{z_1})},$$

where $z_1 = -2+i$ and note that z_1 is a simple pole of $f(z)\cdot e^{iz}$ which lies above the real axis, with residue

$$B_1 = \frac{e^{iz_1}}{z_1-\overline{z_1}} = \frac{1}{2i}e^{-2i-1} = \frac{1}{e}\cdot\frac{e^{-2i}}{2i}.$$

Therefore, when $R>\sqrt{5}$, and C_R denotes the upper half of the positively oriented circle $|z|=R$, we have

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+4x+4}\,dx + \int_{C_R} f(z)e^{iz}\,dz = 2\pi i B_1,$$

and therefore

$$\int_{-R}^{R} \frac{\sin x}{x^2+4x+4}\,dx = \mathrm{Im}(2\pi i B_1) - \mathrm{Im}\int_{C_R} f(z)e^{-z}\,dz. \qquad (*****)$$

Now, on C_R, we have $z = Re^{i\theta}$, $0 \le \theta \le \pi$, so that $z - z_1 = Re^{i\theta} + 2 - i$, and from the back end of the triangle inequality, we have

$$|z - z_1| \ge R - \sqrt{5} \qquad \text{and} \qquad |z - \overline{z_1}| \ge R - \sqrt{5}$$

for $R > \sqrt{5}$, and therefore

$$|f(z)| = \left|\frac{1}{(z - z_1)(z - \overline{z_1})}\right| \le \frac{1}{|z - z_1|^2} \le \frac{1}{(R - \sqrt{5})^2},$$

on C_R, and $M_R = \dfrac{1}{(R - \sqrt{5})^2} \to 0$ as $R \to \infty$.

From Jordan's Lemma, we have

$$\left|\text{Im} \int_{C_R} f(z)e^{-z}\, dz\right| \le \left|\int_{C_R} f(z)e^{-z}\, dz\right| \to 0$$

as $R \to \infty$. Letting $R \to \infty$ in $(*****)$ we have

$$P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x^2 + 4x + 4}\, dx = \text{Im}(2\pi i B_1) = -\frac{\pi}{e} \sin 2.$$

Question 6. Evaluate the improper integral

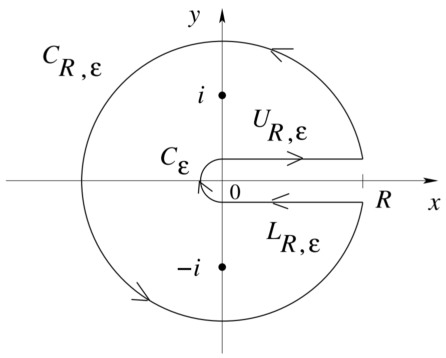
$$\int_0^{\infty} \frac{x^a}{(x^2 + 1)^2}\, dx, \qquad \text{where} \quad -1 < a < 3 \quad \text{and} \quad x^a = \exp(a \ln x).$$

Ans: $\dfrac{(1 - a)\pi}{4\cos(a\pi/2)}$.

Solution: Let $f(z) = \dfrac{z^a}{(z^2 + 1)^2}$, and note that if a is not an integer, then this function is multiple valued, and we will choose the branch given by

$$z^a = e^{a \log z},$$

where $\log z = \ln r + i\theta$, where $r > 0$, and $0 \le \theta < 2\pi$. In order to apply the residue theorem, the contour of integration can only enclose isolated singular points of f, and so it cannot enclose the branch cut $\{z \in \mathbb{C} : z = \text{Re } z \ge 0\}$, so for $0 < \epsilon < 1 < R$, we integrate over the contour below.



Since $0 < \epsilon < 1 < R$, then the isolated singular points $z = \pm i$ are inside the contour, but the branch cut is outside the contour, and applying the residue theorem we have

$$\oint_C \frac{z^a}{(z^2+1)^2}\, dz = 2\pi i \left[\operatorname*{Res}_{z=i}(f(z)) + \operatorname*{Res}_{z=-i}(f(z)) \right] \tag{$\dagger$}$$

where the contour $C = C_{R,\epsilon} + L_{R,\epsilon} + C_\epsilon + U_{R,\epsilon}$ is traversed in the counterclockwise direction.

Now $f(z) = \dfrac{z^a}{(z^2+1)^2}$ has a pole of order $m = 2$ at $z = \pm i$, and

$$\operatorname*{Res}_{z=i}(f(z)) = \frac{d}{dz}\left[(z-i)^2 \frac{z^a}{(z^2+1)^2} \right]\Bigg|_{z=i} = \frac{d}{dz}\left[\frac{e^{a\log z}}{(z+i)^2} \right]\Bigg|_{z=i} = \frac{i(a-1)}{4} e^{i\pi a/2}$$

and

$$\operatorname*{Res}_{z=-i}(f(z)) = \frac{d}{dz}\left[(z+i)^2 \frac{z^a}{(z^2+1)^2} \right]\Bigg|_{z=-i} = \frac{d}{dz}\left[\frac{e^{a\log z}}{(z-i)^2} \right]\Bigg|_{z=-i} = -\frac{i(a-1)}{4} e^{i3\pi a/2},$$

and from (†) we have

$$\oint_C \frac{z^a}{(z^2+1)^2}\, dz = 2\pi i \left[\frac{i(a-1)}{4} \left(e^{i\pi a/2} - e^{i3\pi a/2} \right) \right] = \frac{\pi}{2}(1-a)\left(e^{i\pi a/2} - e^{i3\pi a/2} \right).$$

Now, on C_ϵ, we have

$$\left| \int_{C_\epsilon} \frac{z^a}{(z^2+1)^2}\, dz \right| \le \frac{\pi \epsilon^{1+a}}{(1-\epsilon^2)^2} \to 0$$

as $\epsilon \to 0$ if and only if $a > -1$.

On $C_{R,\epsilon}$, we have

$$\left| \int_{C_{R,\epsilon}} \frac{z^a}{(z^2+1)^2}\, dz \right| \le \frac{2\pi R^{a+1}}{(R^2-1)^2} = \frac{2\pi R^{a-3}}{(1-1/R^2)^2} \to 0$$

as $R \to \infty$ if and only if $a < 3$.

Letting $R \to \infty$ and $\epsilon \to 0$ in (†), we have

$$\lim_{R\to\infty} \left[\int_0^R \frac{e^{a(\ln x + i0)}}{(x^2+1)^2}\, dx + \int_R^0 \frac{e^{a(\ln x + i2\pi)}}{(x^2+1)^2}\, dx \right] = \frac{\pi}{2}(1-a)\left(e^{i\pi a/2} - e^{i3\pi a/2} \right),$$

so that

$$\left(1 - e^{2\pi i a}\right) \int_0^\infty \frac{x^a}{(x^2+1)^2}\, dx = \frac{\pi}{2}(1-a)\left(e^{i\pi a/2} - e^{i3\pi a/2} \right),$$

and since $e^{2\pi i a} \ne 1$, then

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\, dx = \frac{\pi}{2}(1-a)\frac{e^{i\pi a/2} - e^{i3\pi a/2}}{1 - e^{2\pi i a}} = \frac{\pi}{2}(1-a)\frac{e^{-i\pi a/2} - e^{i\pi a/2}}{e^{-i\pi a} - e^{i\pi a}},$$

that is,

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\, dx = \frac{\pi}{2}(1-a)\frac{\sin(\pi a/2)}{\sin(\pi a)} = \frac{(1-a)\pi}{4\cos(\pi a/2)}$$

from the double angle formula.

Question 7. Use residues to evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}.$$

Ans: $\sqrt{2}\,\pi$.

SOLUTION:

On the circle $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, we have

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta,$$

and

$$\sin\theta = \frac{z - z^{-1}}{2i}, \qquad \cos\theta = \frac{z + z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz}$$

so we can rewrite the integral as a contour integral

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} &= \oint_{|z|=1} \frac{1}{\left(1 + \left(\dfrac{z - z^{-1}}{2i}\right)^2\right)} \frac{dz}{iz} \\
&= \frac{1}{i}\oint_{|z|=1} \frac{1}{\left(1 - \frac{1}{4}(z^2 - 2 + 1/z^2)\right)\cdot z}\,dz \\
&= -\frac{4}{i}\oint_{|z|=1} \frac{z\,dz}{z^4 - 6z^2 + 1},
\end{aligned}$$

that is,

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = -\frac{4}{i}\oint_{|z|=1} \frac{z\,dz}{z^4 - 6z^2 + 1}.$$

Note that $z^4 - 6z^2 + 1 = 0$ if and only if $(z^2-3)^2 = 8$, that is, if and only if $z^2 = 3 \pm 2\sqrt{2}$, so that the integrand has four simple poles

$$z_1 = \sqrt{3 + 2\sqrt{2}}, \qquad z_2 = -\sqrt{3 + 2\sqrt{2}}, \qquad z_3 = \sqrt{3 - 2\sqrt{2}}, \qquad z_4 = -\sqrt{3 - 2\sqrt{2}},$$

but $|z_1| > 1$ and $|z_2| > 1$, while $|z_3| < 1$ and $|z_4| < 1$, and only z_3 and z_4 lie inside the contour $C : |z| = 1$.

The residues of the integrand at these poles are

$$\operatorname*{Res}_{z=z_3}(f(z)) = \frac{z_3}{\left(z_3^2 - (3 + 2\sqrt{2})\right)(2z_3)} = -\frac{1}{8\sqrt{2}}$$

and

$$\operatorname*{Res}_{z=z_4}(f(z)) = \frac{z_4}{\left(z_4^2 - (3 + 2\sqrt{2})\right)(2z_4)} = -\frac{1}{8\sqrt{2}},$$

so that

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = 2\pi i\left[-\frac{4}{i}\left(-\frac{1}{8\sqrt{2}} - \frac{1}{8\sqrt{2}}\right)\right] = 8\pi\cdot\frac{1}{4\sqrt{2}} = \sqrt{2}\,\pi.$$

Question 8. Use residues to evaluate the definite integral

$$\int_0^\pi \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2} \qquad (-1 < a < 1).$$

Ans: $\dfrac{a^2\pi}{1-a^2}$.

Solution:

Since $\displaystyle\int_0^\pi \cos 2\theta \, d\theta = 0$, we may assume that $0 < |a| < 1$, and consider the real part of

$$\int_{-\pi}^\pi \frac{e^{2i\theta}}{1 - 2a\cos\theta + a^2} \, d\theta.$$

We convert this to a contour integral around the circle $|z| = 1$ using $dz = ie^{i\theta} d\theta = iz \, d\theta$, and

$$\sin\theta = \frac{z - z^{-1}}{2i}, \qquad \cos\theta = \frac{z + z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz},$$

to get

$$\begin{aligned}
\int_{-\pi}^\pi \frac{e^{2i\theta}}{1 - 2a\cos\theta + a^2} \, d\theta &= \oint_{|z|=1} \frac{z^2}{\left[1 - 2a\left(\dfrac{z+z^{-1}}{2}\right) + a^2\right]} \frac{dz}{iz} \\
&= \frac{1}{i} \oint_{|z|=1} \frac{z^2}{[-az^2 + (1+a^2)z - a]} \, dz \\
&= -\frac{1}{i} \oint_{|z|=1} \frac{z^2}{(z-a)(az-1)} \, dz \\
&= -\frac{1}{ai} \oint_{|z|=1} \frac{z^2}{(z-a)(z-1/a)} \, dz,
\end{aligned}$$

that is,

$$\int_{-\pi}^\pi \frac{e^{2i\theta}}{1 - 2a\cos\theta + a^2} \, d\theta = -\frac{1}{ai} \oint_{|z|=1} \frac{z^2}{(z-a)(z-1/a)} \, dz.$$

The integrand has a simple pole at $z_1 = a$ and $z_2 = \dfrac{1}{a}$, and since $|a| < 1$, only z_1 is inside the contour $|z| = 1$. The residue at z_1 is

$$\operatorname*{Res}_{z=z_1} \left(\frac{z^2}{(z-a)(z-1/a)} \right) = \frac{a^2}{a - 1/a} = -\frac{a^3}{1-a^2},$$

so that

$$\int_{-\pi}^\pi \frac{e^{2i\theta}}{1 - 2a\cos\theta + a^2} \, d\theta = 2\pi i \left(-\frac{1}{ai}\right)\left(-\frac{a^3}{1-a^2}\right) = \frac{2\pi a^2}{1-a^2},$$

and equating real and imaginary parts, we have

$$\int_{-\pi}^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} \, d\theta = \frac{2\pi a^2}{1-a^2},$$

and therefore

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} \, d\theta = \frac{\pi a^2}{1-a^2}.$$

Question 9. Use residues to evaluate the definite integral

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} \quad (a>1).$$

Ans: $\dfrac{a\pi}{\left(\sqrt{a^2-1}\right)^3}$.

Solution: Since

$$\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = 2\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2},$$

we convert this to a contour integral around the circle $|z|=1$ using $dz = ie^{i\theta}d\theta = iz\,d\theta$, and

$$\sin\theta = \frac{z-z^{-1}}{2i}, \qquad \cos\theta = \frac{z+z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz},$$

to get

$$\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \oint_{|z|=1} \frac{1}{\left(a+\dfrac{z+1/z}{2}\right)^2}\frac{dz}{iz} = \frac{4}{i}\oint_{|z|=1} \frac{z}{(z^2+2az+1)^2}\,dz.$$

The integrand has a pole of order 2 at $z_1 = -a+\sqrt{a^2-1}$ and a pole of order 2 at $z_2 = -a-\sqrt{a^2-1}$, however $|z_2|>1$ since $a>1$, while $|z_1| = a-\sqrt{a^2-1}<1$, and z_1 is the only singular point inside the contour $C: \ |z|=1$.

To find the residue at $z=z_1$, we write

$$f(z) = \frac{z}{(z^2+2az+1)^2} = \frac{z}{(z+a-\sqrt{a^2-1})^2(z+a+\sqrt{a^2-1})^2},$$

so that

$$(z-z_1)^2 f(z) = \frac{z}{(z+a+\sqrt{a^2-1})^2},$$

and

$$\frac{d}{dz}\left\{(z-z_1)^2 f(z)\right\} = \frac{-z+a+\sqrt{a^2-1}}{(z+a+\sqrt{a^2-1})^3}.$$

Therefore,

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1}\frac{d}{dz}\left\{(z-z_1)^2 f(z)\right\} = \frac{2a}{\left(2\sqrt{a^2-1}\right)^3} = \frac{a}{4\left(\sqrt{a^2-1}\right)^3},$$

and

$$\oint_{|z|=1} \frac{z}{(z^2+2az+1)^2}\,dz = \frac{2\pi i a}{4\left(\sqrt{a^2-1}\right)^3},$$

that is,

$$\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{4}{i}\cdot\frac{2\pi i a}{4\left(\sqrt{a^2-1}\right)^3} = \frac{2\pi i a}{\left(\sqrt{a^2-1}\right)^3},$$

and finally,

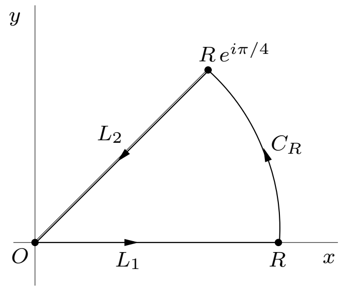
$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{\left(\sqrt{a^2-1}\right)^3}.$$

Question 10.

Evaluate the *Fresnel integrals*:

$$\int_0^\infty \cos(x^2)\,dx \qquad \text{and} \qquad \int_0^\infty \sin(x^2)\,dx$$

by integrating $f(z) = e^{iz^2}$ around the boundary $\mathcal{C}$ of the sector $0 \le r \le R$, and $0 \le \theta \le \pi/4$ shown below and using the Cauch y-Goursat theorem.



Ans: $\sqrt{\dfrac{\pi}{8}}$.

SOLUTION: The function $f(z) = e^{iz^2}$ is analytic inside and on the contour $\mathcal{C}$, and by the Cauchy-Goursat theorem we have

$$0 = \oint_{\mathcal{C}} e^{iz^2}\,dz = \int_{L_1} e^{iz^2}\,dz + \int_{C_R} e^{iz^2}\,dz + \int_{L_2} e^{iz^2}\,dz. \qquad (*)$$

• On L_1: $z = x$, $0 \le x \le R$, $dz = dx$, so that

$$\int_{L_1} e^{iz^2}\,dz = \int_0^R e^{ix^2}\,dx = \int_0^R \cos(x^2)\,dx + i\int_0^R \sin(x^2)\,dx.$$

• On C_R: $z = Re^{i\theta}$, $0 \le \theta \le \pi/4$, $dz = iRe^{i\theta}\,d\theta$, so that

$$\int_{C_R} e^{iz^2}\,dz = \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2 \sin 2\theta} iRe^{i\theta}\,d\theta = iR\int_0^{\pi/4} e^{i(R^2\cos 2\theta + \theta)} e^{-R^2\sin 2\theta}\,d\theta,$$

and by Jordan's inequality

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta = \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi < \frac{\pi}{2R}\left(1 - e^{-R^2}\right).$$

Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} e^{iz^2}\,dz\right| = 0.$$

• On L_2: $z = re^{i\pi/4}$, $dz = e^{i\pi/4}dr$, and

$$\int_{L_2} e^{iz^2}\,dz = \int_R^0 e^{ir^2 e^{i\pi/2}} e^{i\pi/4}\,dr = -e^{i\pi/4}\int_0^R e^{ir^2(cos\pi/2 + i\sin\pi/2)}\,dr = -e^{i\pi/4}\int_0^R e^{-r^2}\,dr.$$

Letting $R \to \infty$ in $(*)$, we have

$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx = e^{i\pi/4}\int_0^\infty e^{-r^2}\,dr = e^{i\pi/4}\sqrt{\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\sqrt{\frac{\pi}{4}},$$

thus,

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \sqrt{\frac{\pi}{8}}.$$

Note that

$$\left(\int_0^\infty e^{-r^2}\,dr\right)^2 = \int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy = \int_0^{\pi/2} \left(\int_0^\infty e^{-r^2} r dr\right) d\theta,$$

and integrating,

$$\int_0^\infty e^{-r^2}\,dr = \sqrt{\frac{\pi}{4}}.$$