Math 309 - Spring-Summer 2017

Solutions to Problem Set # 11

Completion Date: Friday July 28, 2017

Question 1.

Find the residue at $z = 0$ of the function

$$\text{(a)}\ \frac{1}{z+z^2}; \qquad \text{(b)}\ z\cos\left(\frac{1}{z}\right); \qquad \text{(c)}\ \frac{z-\sin z}{z}; \qquad \text{(d)}\ \frac{\cot z}{z^4}; \qquad \text{(e)}\ \frac{\sinh z}{z^4(1-z^2)}.$$

Ans: (a) 1; (b) $-1/2$; (c) 0; (d) $-1/45$; (e) $7/6$.

SOLUTION:

(a) For $0 < |z| < 1$, we have

$$\frac{1}{z+z^2} = \frac{1}{z}\cdot\frac{1}{1+z} = \frac{1}{z}\left\{1 - z + z^2 - z^3 + \cdots\right\} = \frac{1}{z} - 1 + z - z^2 + \cdots,$$

so the residue at $z = 0$ is 1.

(b) For $z \neq 0$, we have

$$z\cos\left(\frac{1}{z}\right) = z\left\{1 - \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^4} - + \cdots\right\} = z - \frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{4!}\cdot\frac{1}{z^3} - + \cdots,$$

so the residue at $z = 0$ is $-\dfrac{1}{2!}$.

(c) For $z \neq 0$, we have

$$\frac{z-\sin z}{z} = \frac{1}{z}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots\right)\right\} = \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots,$$

and there are **no** negative powers of z, so the residue at $z = 0$ is 0.

(d) For $0 < |z| < \pi$, we have

$$\frac{\cot z}{z^4} = \frac{\cos z}{z^4\sin z} = \frac{1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots}{z^4\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - + \cdots\right)}.$$

Now let

$$w = \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - + \cdots,$$

then for $z \neq 0$ such that $|w| < 1$, we have

$$\frac{\cot z}{z^4} = \frac{1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots}{z^5(1-w)} = \frac{1}{z^5}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\right)\left(1 + w + w^2 + w^3 + \cdots\right).$$

Therefore, for $0 < |z| < \pi$, we have

$$\frac{\cot z}{z^4} = \frac{1}{z^5}\big(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\big)\big(1 + \frac{z^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 + \cdots\big),$$

that is,

$$\frac{\cot z}{z^4} = \frac{1}{z^5} - \left(\frac{1}{2!} - \frac{1}{3!}\right) \cdot \frac{1}{z^3} + \left[\frac{1}{(3!)^2} - \frac{1}{5!} + \frac{1}{4!} - \frac{1}{2!3!}\right] \cdot \frac{1}{z} + \cdots$$

for $0 < |z| < \pi$, and

$$\operatorname*{Res}_{z=0}\left(\frac{\cot z}{z^4}\right) = \frac{1}{(3!)^2} - \frac{1}{5!} + \frac{1}{4!} - \frac{1}{2!3!} = -\frac{1}{45}.$$

(e) For $0 < |z| < 1$, we have

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3}\big(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\big)\big(1 + z^2 + z^4 + \cdots\big),$$

so that

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3}\left\{1 + \left(1 + \frac{1}{3!}\right) z^2 + \cdots + \text{ higher order terms }\right\},$$

that is,

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3} + \frac{7}{6} \cdot \frac{1}{z} + \cdots$$

for $0 < |z| < 1$, and

$$\operatorname*{Res}_{z=0}\left(\frac{\sinh z}{z^4(1-z^2)}\right) = \frac{7}{6}.$$

Question 2.

Use Cauchy's residue theorem to evaluate the integral of

$$\frac{\exp(-z)}{z^2}$$

around the circle $|z| = 3$ in the positive sense.

Ans: $-2\pi i$.

Solution: The function $\dfrac{e^{-z}}{z^2}$ has an isolated singularity at $z = 0$ which is inside the circle $|z| = 3$, and since

$$\frac{e^{-z}}{z^2} = \frac{1}{z^2} \cdot \left\{1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right\} = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots$$

for $0 < |z| < \infty$, then

$$\operatorname*{Res}_{z=0}\left(\frac{e^{-z}}{z^2}\right) = -1.$$

Now, since $\dfrac{e^{-z}}{z^2}$ is analytic inside and on $|z| = 3$, except at $z = 0$, then

$$\oint_{|z|=3} \frac{e^{-z}}{z^2}\, dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(\frac{e^{-z}}{z^2}\right) = -2\pi i.$$

Question 3.

Use Cauchy's residue theorem to evaluate the integral of

$$z^2 \exp\left(\frac{1}{z}\right)$$

around the circle $|z| = 3$ in the positive sense.

Ans: $\pi i/3$.

SOLUTION: The function $z^2 \exp\left(\frac{1}{z}\right)$ has an isolated (essential) singularity at $z = 0$ which is inside the circle $|z| = 3$, and since

$$z^2 \exp\left(\frac{1}{z}\right) = z^2 \left\{1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \frac{1}{4!} \cdot \frac{1}{z^4} + \cdots\right\}$$

for $0 < |z| < \infty$, then

$$\operatorname*{Res}_{z=0}\left(z^2 \exp\left(\frac{1}{z}\right)\right) = \frac{1}{3!}.$$

Now, since $z^2 \exp\left(\frac{1}{z}\right)$ is analytic inside and on $|z| = 3$, except at $z = 0$, then

$$\oint_{|z|=3} z^2 \exp\left(\frac{1}{z}\right) dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(z^2 \exp\left(\frac{1}{z}\right)\right) = 2\pi i \cdot \frac{1}{3!} = \frac{\pi i}{3}.$$

Question 4.

Use Cauchy's residue theorem to evaluate the integral of

$$\frac{z+1}{z^2 - 2z}$$

around the circle $|z| = 3$ in the positive sense.

Ans: $2\pi i$.

SOLUTION: The function $\frac{z+1}{z^2 - 2z} = \frac{z+1}{z(z-2)}$ has isolated singularities at $z = 0$ and $z = 2$, both of which lie inside the circle $|z| = 3$, so that

$$\oint_{|z|=3} \frac{z+1}{z^2 - 2z}\, dz = 2\pi i \left\{\operatorname*{Res}_{z=0}\left(\frac{z+1}{z^2 - 2z}\right) + \operatorname*{Res}_{z=2}\left(\frac{z+1}{z^2 - 2z}\right)\right\}$$

Now, since

$$\frac{z+1}{z(z-2)} = \frac{1}{z-2} + \frac{1}{z(z-2)} = \frac{3}{2} \cdot \frac{1}{z-2} - \frac{1}{2} \cdot \frac{1}{z},$$

and since $\frac{1}{z-2}$ is analytic at $z = 0$, then

$$\operatorname*{Res}_{z=0}\left(\frac{z+1}{z^2 - 2z}\right) = -\frac{1}{2},$$

while since $\frac{1}{z}$ is analytic at $z = 2$, then

$$\operatorname*{Res}_{z=2}\left(\frac{z+1}{z^2 - 2z}\right) = \frac{3}{2},$$

and

$$\oint_{|z|=3} \frac{z+1}{z^2-2z}\,dz = 2\pi i\left[-\frac{1}{2}+\frac{3}{2}\right] = 2\pi i.$$

Question 5.

In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $z\exp\left(\frac{1}{z}\right)$; (b) $\frac{z^2}{1+z}$; (c) $\frac{\sin z}{z}$; (d) $\frac{\cos z}{z}$; (e) $\frac{1}{(2-z)^3}$.

SOUTION:

(a) For $0<|z|<\infty$, we have

$$z\exp\left(\frac{1}{z}\right) = z + 1 + \underbrace{\frac{1}{2!}\cdot\frac{1}{z} + \frac{1}{3!}\cdot\frac{1}{z^2} + \cdots + \frac{1}{n!}\cdot\frac{1}{z^{n-1}} + \cdots}_{\text{principal part}}$$

and the isolated singular point $z=0$ is an **essential singular point** since the principal part has infinitely many nonzero terms.

(b) For $z \neq -1$, we have

$$\frac{z^2}{z+1} = 1 + z - 2 + \underbrace{\frac{1}{1+z}}_{\text{principal part}}$$

the isolated singular point $z=-1$ is a **simple pole**, that is, a **pole of order 1**.

(c) For $0<|z|<\infty$, we have

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots,$$

and the isolated singular point $z=0$ is a **removable singular point**, since there are **no** nonzero terms in the principal part.

(d) For $0<|z|<\infty$, we have

$$\frac{\cos z}{z} = \underbrace{\frac{1}{z}}_{\text{principal part}} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots$$

and the isolated singular point $z=0$ is a **simple pole**, that is, a **pole of order 1**.

(e) For $z\neq 2$, we have

$$\frac{1}{(2-z)^3} = -\underbrace{\frac{1}{(z-2)^3}}_{\text{principal part}}$$

and the isolated singular point $z=2$ is a **pole of order 3**.

Question 6.

Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; (b) $\dfrac{1-\exp(2z)}{z^4}$; (c) $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans: (a) $m=1,\ B=-1/2$; (b) $m=3,\ B=-4/3$; (c) $m=2,\ B=2e^2$.

SOLUTION:

(a) For $z \neq 0$ we have

$$\frac{1-\cosh z}{z^3} = \frac{1}{z^3}\left\{1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)\right\} = -\frac{1}{2!}\cdot\frac{1}{z}-\frac{z}{4!}-\frac{z^3}{6!}-\cdots,$$

and the isolated singularity $z=0$ is a **simple pole**, that is, a **pole of order 1**, with residue $B=-\dfrac{1}{2!}$.

(b) For $z \neq 0$, we have

$$\frac{1-e^{2z}}{z^4} = \frac{1}{z^4}\left\{1-\left(1+\frac{2z}{1!}+\frac{2^2z^2}{2!}+\frac{2^3z^3}{3!}+\frac{2^4z^4}{4!}+\cdots\right)\right\} = -2\cdot\frac{1}{z^3}-2\cdot\frac{1}{z^2}-\frac{4}{3}\cdot\frac{1}{z}-\frac{2}{3}-\frac{4}{15}z-\cdots,$$

and the isolated singular point $z=0$ is a **pole of order 3**, with residue $B=-\dfrac{4}{3}$.

(c) For $z \neq 1$, we have

$$\frac{e^{2z}}{(z-1)^2} = \frac{e^2}{(z-1)^2}\cdot e^{2(z-1)} = \frac{e^2}{(z-1)^2}\left\{1+2(z-1)+\frac{2^2(z-1)^2}{2!}+\frac{2^3(z-1)^3}{3!}+\cdots\right\}$$

and the isolated singular point $z=1$ is a **pole of order 2** with residue $B=2e^2$.

Question 7.

In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z^2+2}{z-1}$; (b) $\left(\dfrac{z}{2z+1}\right)^3$; (c) $\dfrac{\exp(z)}{z^2+\pi^2}$.

Ans: (a) $m=1,\ B=3$; (b) $m=3,\ B=-3/16$; (c) $m=1,\ B=\pm i/2\pi$.

SOLUTION:

(a) Note that

$$\frac{z^2+2}{z-1} = \frac{\Phi(z)}{z-1}$$

where $\Phi(z)=z^2+2$ is analytic at $z=1$, and $\Phi(1)=3\neq 0$, so that $\dfrac{z^2+2}{z-1}$ has a simple pole at $z=1$ so that $m=1$ and

$$B = \operatorname*{Res}_{z=1}\left(\frac{z^2+2}{z-1}\right) = \Phi(1) = 3.$$

(b) Note that

$$\left(\frac{z}{2z+1}\right)^3 = \frac{z^3}{8} \cdot \frac{1}{\left(z+\dfrac{1}{2}\right)^3} = \frac{\Phi(z)}{\left(z+\dfrac{1}{2}\right)^3}$$

where $\Phi(z) = \dfrac{z^3}{8}$ is analytic at $z = -\dfrac{1}{2}$ and $\Phi\left(-\dfrac{1}{2}\right) = -\dfrac{1}{64} \neq 0$, so that $\left(\dfrac{z}{2z+1}\right)^3$ has a pole of order 3 at $z = -\dfrac{1}{2}$, so that $m = 3$ and

$$B = \operatorname*{Res}_{z=-1/2}\left(\frac{z^3}{(2z+1)^3}\right) = \frac{\Phi''(-1/2)}{2!} = \frac{1}{2!} \cdot \frac{1}{8} \cdot 3 \cdot 2 \cdot (-1/2) = -\frac{3}{16}.$$

(c) Note that

$$\frac{e^z}{z^2+\pi^2} = \frac{\Phi_1(z)}{z - \pi i}$$

where

$$\Phi_1(z) = \frac{e^z}{z+\pi i}$$

is analytic at $z = \pi i$, and $\Phi_1(\pi i) = \dfrac{e^{\pi i}}{2\pi i} \neq 0$, so that $\dfrac{e^z}{z^2+\pi^2}$ has a simple pole at $z = \pi i$ and

$$B_1 = \operatorname*{Res}_{z=\pi i}\left(\frac{e^z}{z^2+\pi^2}\right) = \frac{e^{\pi i}}{2\pi i} = -\frac{1}{2\pi i} = \frac{i}{2\pi}.$$

Also,

$$\frac{e^z}{z^2+\pi^2} = \frac{\Phi_2(z)}{z + \pi i}$$

where

$$\Phi_2(z) = \frac{e^z}{z-\pi i}$$

is analytic at $z = -\pi i$, and $\Phi_2(-\pi i) = \dfrac{e^{-\pi i}}{-2\pi i} \neq 0$, so that $\dfrac{e^z}{z^2+\pi^2}$ has a simple pole at $z = -\pi i$ and

$$B_2 = \operatorname*{Res}_{z=-\pi i}\left(\frac{e^z}{z^2+\pi^2}\right) = \frac{e^{-\pi i}}{-2\pi i} = -\frac{1}{-2\pi i} = -\frac{i}{2\pi}.$$

Question 8.

Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz,$$

taken counterclockwise around the circle

(a) $|z-2| = 2$; (b) $|z| = 4$.

Ans: (a) πi; (b) $6\pi i$.

SOLUTION:

(a) Note that

$$f(z) = \frac{3z^3+2}{(z-1)(z^2+9)} = \frac{\Phi(z)}{z-1}$$

where

$$\Phi(z) = \frac{3z^3+2}{z^2+9}$$

is analytic at $z = 1$ and $\Phi(1) = \dfrac{5}{10} = \dfrac{1}{2} \neq 0$, so that f has simple pole at $z = 1$ with residue $\dfrac{1}{2}$.
Since f is analytic inside and on the circle $C: \ |z-2| = 2$, except at $z = 1$, which is inside C, then

$$\oint_{|z-2|=2} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i \cdot \operatorname*{Res}_{z=1}\left(\frac{3z^3+2}{(z-1)(z^2+9)}\right) = 2\pi i \cdot \frac{1}{2} = \pi i.$$

(b) Now the singular points of

$$f(z) = \frac{3z^3+2}{(z-1)(z^2+9)}$$

are $z = 1$, $z = 3i$, $z = -3i$ and are all **inside** the circle $C: \ |z| = 4$, and (check this!) since f has **simple poles** at $z = 1, \pm 3i$, and since

$$\operatorname*{Res}_{z=1}(f(z)) = \lim_{z\to 1}(z-1)f(z) = \frac{1}{2}$$

$$\operatorname*{Res}_{z=3i}(f(z)) = \lim_{z\to 3i}(z-3i)f(z) = \frac{2-81i}{6i(3i-1)}$$

$$\operatorname*{Res}_{z=-3i}(f(z)) = \lim_{z\to -3i}(z+3i)f(z) = \frac{2+81i}{6i(3i+1)}$$

then

$$\oint_{|z|=4} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz = 2\pi i\left\{\frac{1}{2} + \frac{2-81i}{6i(3i-1)} + \frac{2+81i}{6i(3i+1)}\right\} = 2\pi i \cdot 3 = 6\pi i.$$

Question 9.

Show that

(a) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi}$;

(b) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos \pi t.$

Solution:

(a) Note that

$$f(z) = \frac{p(z)}{q(z)}$$

where

$$p(z) = z - \sinh z \qquad \text{and} \qquad q(z) = z^2 \sinh z$$

are analytic at $z = \pi i$.
Now, the zeros of $\sinh z$ are at $z = n\pi i$ for $n = 0, \pm 1, \pm 2, \ldots$, and

$$p(\pi i) = \pi i - \sinh(\pi i) = \pi i \neq 0,$$

while

$$q(\pi i) = (\pi i)^2 \sinh(\pi i) = 0,$$

but

$$q'(\pi i) = 2\pi i \sinh(\pi i) + (\pi i)^2 \cosh(\pi i) = -\pi^2 \cos \pi = \pi^2 \neq 0,$$

so that $q(z)$ has a simple zero at $z = \pi i$.

Therefore $f(z) = \dfrac{p(z)}{q(z)}$ has a simple pole at $z = \pi i$, and

$$B = \operatorname*{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}.$$

(b) Note that the function

$$f(z) = \frac{e^{zt}}{\sinh z}$$

is analytic except where $\sinh z = 0$, that is, except at $z = n\pi i,\ n = 0, \pm 1, \pm 2, \ldots$.

Now,

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{e^{zt}}{\left(\dfrac{\sinh z}{z}\right)} = 1,$$

and f has a simple pole at the isolated singularity $z = 0$ with $\operatorname*{Res}_{z=0}(f(z)) = 1$.

Also, since $\sinh(z - n\pi i) = (-1)^n \sinh z$, then

$$\lim_{z\to n\pi i} (z - n\pi i) f(z) = \lim_{z\to n\pi i} \frac{(-1)^n e^{zt}}{\left(\dfrac{\sinh(z - n\pi i)}{z - n\pi i}\right)} = (-1)^n e^{n\pi i t},$$

and f has a simple pole at the isolated singularity $z = n\pi i$ for each $n = \pm 1, \pm 2, \ldots$ with residue $(-1)^n e^{n\pi i t}$.

Therefore,

$$\operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -e^{\pi i t} - e^{-\pi i t} = -2\cos \pi t$$

from Euler's formula.