Math 309 - Spring-Summer 2017

Solutions to Problem Set # 10

Completion Date: Friday July 14, 2017

Question 1.

By differentiating the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

obtain the expansions

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \qquad (|z| < 1).$$

Solution: Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$, differentiating the right-hand side term by term, we have

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{m=0}^{\infty} (m+1) z^m$$

for $|z| < 1$.

Differentiating again, we have

$$\frac{2}{(1-z)^3} = \frac{d}{dz}\left(\frac{1}{(1-z)^2}\right) = \sum_{m=0}^{\infty} (m+1)\cdot m z^{m-1} = \sum_{m=1}^{\infty} (m+1)\cdot m z^{m-1} = \sum_{n=0}^{\infty} (n+2)(n+1) z^n$$

for $|z| < 1$.

Question 2.

By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad (|z| < 1),$$

found in Question 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty).$$

Solution: Since

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$$

for $|z| < 1$, replacing z by $\dfrac{1}{1-z}$ in this expression, we have

$$\frac{1}{\left(1 - \dfrac{1}{1-z}\right)^2} = \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{1-z}\right)^n,$$

that is,

$$\frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^n}$$

for $\left|\dfrac{1}{z-1}\right| < 1$, that is, for $|z-1| > 1$.

Therefore,

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^{n+2}} = \sum_{m=2}^{\infty} \frac{(m-1)}{(1-z)^m} = \sum_{m=2}^{\infty} \frac{(-1)^m(m-1)}{(z-1)^m}$$

for $1 < |z-1| < \infty$.

Question 3.

Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

Solution: We have

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n(z-2)^n}{2^n}$$

for $|z-2| < 2$.

Differentiating this expression term by term, we have

$$-\frac{1}{z^2} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n n(z-2)^{n-1}}{2^n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n n(z-2)^{n-1}}{2^n} = \frac{1}{2} \cdot \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(m+1)(z-2)^m}{2^{m+1}},$$

for $|z-2| < 2$, that is,

$$\frac{1}{z^2} = \frac{1}{4} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m(m+1)(z-2)^m}{2^m}$$

for $|z-2| < 2$.

Question 4.

With the aid of series, prove that the function f defined by means of the equations

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{when} \quad z \neq 0, \\ 1 & \text{when} \quad z = 0 \end{cases}$$

is entire.

SOLUTION:

The Maclaurin series for e^z is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

and this series converges to e^z for all z, $|z| < \infty$.

If $z \neq 0$, then

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n+1)!} + \cdots,$$

that is, the series converges to $\dfrac{e^z - 1}{z}$ for all $z \neq 0$, while if $z = 0$, the series converges to 1.

Therefore, if

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{when} \quad z \neq 0, \\ 1 & \text{when} \quad z = 0, \end{cases}$$

then

$$f(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n+1)!} + \cdots$$

for all $z \in \mathbb{C}$, and f is analytic at each $z \in \mathbb{C}$, that is, f is entire.

Question 5.

Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}\,z - \frac{5}{6}\,z^2 + \cdots \qquad (0 < |z| < 1).$$

SOLUTION: We have

$$e^z \cdot \frac{1}{z^2+1} = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 - z^2 + z^4 - z^6 + \cdots\right)$$

for $|z| < 1$, that is,

$$e^z \cdot \frac{1}{z^2+1} = 1 + z - \frac{1}{2}z^2 + \left(\frac{1}{6} - 1\right) z^3 + \cdots$$

for $|z| < 1$, that is,

$$e^z \cdot \frac{1}{z^2+1} = 1 + z - \frac{1}{2}z^2 - \frac{5}{6}z^3 + \cdots$$

for $|z| < 1$, and therefore,

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}\,z - \frac{5}{6}\,z^2 + \cdots$$

for $0 < |z| < 1$.

Question 6.

By writing $\csc z = 1/\sin z$ and then using division, show that

$$\csc z = \frac{1}{z} + \frac{1}{3!}\,z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \cdots \qquad (0 < |z| < \pi).$$

Solution: Since $\sin z = 0$ for $z = 0,\ \pm\pi,\ \pm 2\pi,\ \ldots$, then

$$\csc z = \frac{1}{\sin z}$$

is analytic for $0 < |z| < \pi$.

Now, for $0 < |z| < \pi$

$$z\csc z = \frac{z}{z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots} = \frac{1}{1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \dfrac{z^6}{7!} + \cdots}$$

is analytic since the denominator doesn't vanish for $0 < |z| < \pi$; and for $z = 0$, the series converges to 1. Therefore the function

$$g(z) = \begin{cases} z\csc z & 0 < |z| < \pi \\ 1 & z = 0 \end{cases}$$

is analytic in the entire disk $|z| < \pi$, and so has a Maclaurin series expansion

$$g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

for $|z| < \pi$.

Now, for $0 < |z| < \pi$,

$$z = g(z)\cdot \sin z = (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)(z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots),$$

that is,

$$z = a_0 z + a_1 z^2 + \left(a_2 - \frac{a_0}{3!}\right) z^3 + \left(a_3 - \frac{a_1}{3!}\right) z^4 + \left(a_4 - \frac{a_2}{3!} + \frac{a_0}{5!}\right) z^5 + \cdots$$

for $|z| < \pi$.

So we must have

$$a_0 = 1$$

$$a_1 = 0$$

$$a_2 = \frac{1}{3!}$$

$$a_3 = 0$$

$$a_4 = \frac{1}{(3!)^2} - \frac{1}{5!},$$

and

$$g(z) = 1 + \frac{1}{3!}z^2 + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^4 + \cdots$$

for $|z| < \pi$.

Therefore,

$$\csc z = \frac{g(z)}{z} = \frac{1}{z} + \frac{1}{3!}\,z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \cdots$$

for $0 < |z| < \pi$.