Math 309 - Spring-Summer 2017

Solutions to Problem Set # 1

Completion Date: Friday May 12, 2017

Question 1.

Show that

(a) $\text{Re}\,(iz) = -\text{Im}\ z$; (b) $\text{Im}\,(iz) = \text{Re}\ z$.

SOLUTION: If $z = x + iy$, then $iz = -y + ix$, so that

(a) $\text{Re}\,(iz) = -y = -\text{Im}\ z$, and

(b) $\text{Im}\,(iz) = x = \text{Re}\ z$.

Question 2.

Reduce the quantity $\dfrac{5i}{(1-i)(2-i)(3-i)}$ to a real number.

SOLUTION: We have

$$\frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(1-i)(5-5i)} = \frac{i}{(1-i)^2} = \frac{i}{-2i} = -\frac{1}{2}.$$

Question 3.

Reduce the quantity $(1-i)^4$ to a real number.

SOLUTION: We have

$$(1-i)^4 = \left[(1-i)^2\right]^2 = (-2i)^2 = 4i^2 = -4.$$

Question 4.

Verify that $\sqrt{2}\,|z| \geq |\text{Re}\ z| + |\text{Im}\ z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

SOLUTION: Note that

$$0 \leq (|\text{Re}\ z| - |\text{Im}\ z|)^2 = |\text{Re}\ z|^2 - 2|\text{Re}\ z|\,|\text{Im}\ z| + |\text{Im}\ z|^2,$$

so that

$$2|\text{Re}\ z|\,|\text{Im}\ z| \leq |\text{Re}\ z|^2 + |\text{Im}\ z|^2,$$

and

$$|\text{Re}\ z|^2 + 2|\text{Re}\ z|\,|\text{Im}\ z| + |\text{Im}\ z|^2 \leq 2\left(|\text{Re}\ z|^2 + |\text{Im}\ z|^2\right),$$

that is,

$$(|\text{Re } z| + |\text{Im } z|)^2 \le 2\left(|\text{Re } z|^2 + |\text{Im } z|^2\right) = 2|z|^2,$$

and therefore,

$$|\text{Re } z| + |\text{Im } z| \le \sqrt{2}\,|z|.$$

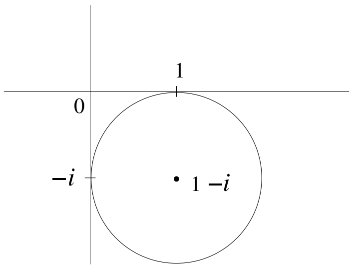
Question 5.

In each case, sketch the set of points determined by the given condition:

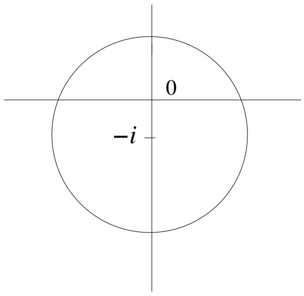
(a) $|z - 1 + i| = 1$; (b) $|z + i| \le 3$; (c) $|z - 4i| \ge 4$.

SOLUTION:

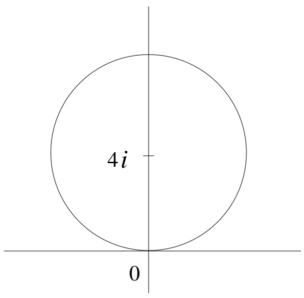
(a) The set $\{z \in \mathbb{C} : |z - 1 + i| = 1\}$ is the circle centered at $1 - i$ with radius 1.



(b) The set $\{z \in \mathbb{C} : |z + i| \le 3\}$ is the closed disk centered at $-i$ with radius 3.



(c) The set $\{z \in \mathbb{C} : |z - 4i| \ge 4\}$ is the set of all points on and outside the circle centered at $4i$ with radius 4.



Question 6.

Use the properties of conjugates and modulii established in class to show that

(a) $\overline{\overline{z} + 3i} = z - 3i$ (b) $\overline{iz} = -i\,\overline{z}$;

(c) $\overline{(2+i)^2} = 3 - 4i$; (d) $\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}\,|2z+5|$.

SOLUTION:

(a) Since $\overline{\overline{z}} = z$, then $\overline{\overline{z} + 3i} = \overline{\overline{z}} + \overline{3i} = z - 3i$.

(b) $\overline{iz} = \overline{-y + ix} = -y - ix = -i(x - iy) = -i\,\overline{z}$.

(c) $\overline{(2+i)^2} = \left(\overline{2+i}\right)^2 = (2-i)^2 = 3 - 4i$.

(d) $\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = |2\overline{z}+5| \cdot |\sqrt{2}-i| = \sqrt{3} \cdot |\overline{2z+5}| = \sqrt{3} \cdot |2z+5|$.

Question 7.

Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|}.$$

SOLUTION: If $|z_3| \neq |z_4|$, then

$$|z_1+z_2| \leq |z_1|+|z_2| \qquad \text{and} \qquad |z_3+z_4| \geq \big||z_3|-|z_4|\big|,$$

so that

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|}.$$

Question 8.

Find the principal argument Arg z when $z = \dfrac{i}{-2-2i}$.

Ans. $-\dfrac{3\pi}{4}$.

SOLUTION: Note that

$$z = \frac{i}{-2-2i} = -\frac{1}{2} \cdot \frac{i}{i+1} = -\frac{1}{2}i\left(\frac{1-i}{2}\right) = -\frac{1}{4}(1+i),$$

that is,

$$z = -\frac{\sqrt{2}}{4}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right].$$

Therefore, $|z| = \dfrac{\sqrt{2}}{4}$ and $\text{Arg}(z) = -\dfrac{3\pi}{4}$.

Question 9.

Using the fact that the modulus $\left|e^{i\theta}-1\right|$ is the distance between the points $e^{i\theta}$ and 1, give a geometric argument to find a value of θ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $\left|e^{i\theta}-1\right| = 2$.

Ans. π.

SOLUTION: Note that

$$|e^{i\theta}-1|^2 = |\cos\theta + i\sin\theta - 1|^2 = (\cos\theta - 1)^2 + \sin^2\theta = 4$$

if and only if $\cos\theta = -1$, that is, if and only if $\theta = \pi$. Geometrically, $|e^{i\theta}-1|$ is the distance between the points $z_1 = e^{i\theta}$ and $z_2 = 1$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and this is a maximum of 2 when $\theta = \pi$.

Question 10.

Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity*:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

SOLUTION: If $z \neq 1$, then

$$\begin{aligned}(1-z)(1 + z + z^2 + \cdots z^n) &= 1 + z + z^2 + \cdots + z^n - (z + z^2 + \cdots + z^{n+1}) \\ &= 1 - z^{n+1},\end{aligned}$$

so that

$$1 + z + z^2 + \cdots z^n = \begin{cases} \dfrac{1 - z^{n+1}}{1 - z} & \text{if } z \neq 1 \\ n+1 & \text{if } z = 1. \end{cases}$$

Taking $z = e^{i\theta}$, where $0 < \theta < 2\pi$, then $z \neq 1$, so that

$$\begin{aligned}1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} &= \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} = \frac{1 - e^{(n+1)i\theta}}{-e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)} \\ &= \frac{-e^{-i\theta/2}\left(1 - e^{(n+1)i\theta}\right)}{2i\sin(\theta/2)} = \frac{i\left(e^{-i\theta/2} - e^{(n+\frac{1}{2})i\theta}\right)}{2\sin(\theta/2)} \\ &= \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)} + \frac{i}{2\sin(\theta/2)}\left(\cos(\theta/2) - \cos\left(n+\tfrac{1}{2}\right)\theta\right)\end{aligned}$$

Equating real and imaginary parts, we have

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)}$$

and

$$\sin\theta + \sin 2\theta + \cdot + \sin n\theta = \frac{1}{2}\cot(\theta/2) - \frac{\cos\left(n+\frac{1}{2}\right)\theta}{2\sin(\theta/2)}.$$