Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers

Complex Variable Evaluation of Dirichlet's Integral

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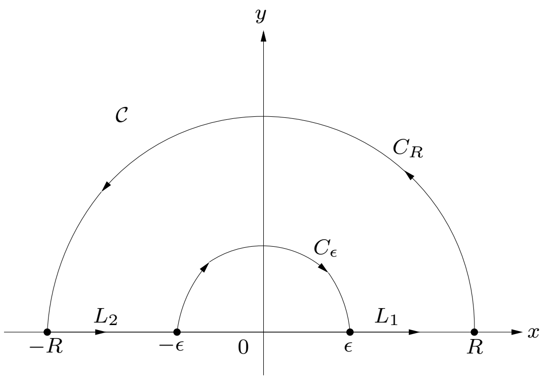
In this note we use the theory of residues to evaluate Dirichlet's integral.

Theorem.

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Proof. We evaluate the integral using the Cauchy-Goursat theorem and integrating the function e^{iz}/z around the indented contour $\mathcal{C}$ shown below, where $0 < \epsilon < 1 < R$.



Since the function e^{iz}/z is analytic inside and on the contour $\mathcal{C}$, by the Cauchy-Goursat theorem

$$0 = \int_{\mathcal{C}} \frac{e^{iz}}{z}\, dz,$$

that is,

$$\int_{L_1} \frac{e^{iz}}{z}\, dz + \int_{C_R} \frac{e^{iz}}{z}\, dz + \int_{L_2} \frac{e^{iz}}{z}\, dz + \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = 0. \qquad (*)$$

- On L_1: $z = x$, where $\epsilon \le x \le R$, and

$$\int_{L_1} \frac{e^{iz}}{z}\, dz = \int_\epsilon^R \frac{e^{ix}}{x}\, dx.$$

- On L_2: $z = x$, where $-R \le x \le -\epsilon$, and

$$\int_{L_2} \frac{e^{iz}}{z}\, dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\, dx = -\int_\epsilon^R \frac{e^{-ix}}{x}\, dx.$$

Therefore,

$$\int_{L_1} \frac{e^{iz}}{z}\, dz + \int_{L_2} \frac{e^{iz}}{z}\, dz = 2i \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx,$$

and

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{L_1} \frac{e^{iz}}{z}\, dz + \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \int_{L_2} \frac{e^{iz}}{z}\, dz = 2i \int_{0}^{\infty} \frac{\sin x}{x}\, dx.$$

• On $C_R:\ z = Re^{i\theta}$, where $0 \le \theta \le \pi$, and

$$\int_{C_R} \frac{e^{iz}}{z}\, dz = \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta}\, d\theta = i \int_0^{\pi} e^{iR\cos\theta}\, e^{-R\sin\theta}\, d\theta.$$

From Jordan's inequality, we have

$$\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| \le \int_0^{\pi} e^{-R\sin\theta}\, d\theta \le \frac{\pi}{R}\left(1 - e^{-R}\right).$$

and therefore,

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, dz = 0.$$

• On $C_\epsilon:\ z = \epsilon e^{i\theta}$, where $0 \le \theta \le \pi$, and the Laurent series expansion of e^{iz}/z about $z = 0$ is

$$\frac{e^{iz}}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(iz)^{n-1}}{n!}$$

valid for $0 < |z| < \infty$, and e^{iz}/z has a simple pole at $z = 0$ with residue 1. The function

$$g(z) = \sum_{n=1}^{\infty} \frac{(iz)^{n-1}}{n!}$$

for $z \in \mathbb{C}$ is an entire function and is continuous and hence bounded on the disk $|z| \le 1$, so there is an $M > 0$ such that $|g(z)| \le M$ for $|z| \le 1$. Therefore,

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = \int_{C_\epsilon} \frac{1}{z}\, dz + \int_{C_\epsilon} g(z)\, dz,$$

and since $0 < \epsilon < 1$, then

$$\left| \int_{c_\epsilon} g(z)\, dz \right| \le M\,\pi\,\epsilon$$

while

$$\int_{C_\epsilon} \frac{1}{z}\, dz = \int_{\pi}^{0} \frac{i\epsilon\, e^{i\theta}}{\epsilon\, e^{i\theta}}\, d\theta = -\int_0^{\pi} i\, d\theta = -\pi i.$$

Therefore,

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = \lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{1}{z}\, dz + \lim_{\epsilon\to 0} \int_{C_\epsilon} g(z)\, dz = -\pi i.$$

Letting $\epsilon \to 0$ and $R \to \infty$ in $(*)$, we get

$$2i \int_0^{\infty} \frac{\sin x}{x}\, dx - \pi i = 0,$$

that is,

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

It is instructive to compare the complex variable proof of this theorem with a proof using real variable techniques.