Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers

Real Variable Evaluation of Dirichlet's Integral

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In this note we will study Dirichet's integral, and give an elementary real variable method for evaluating it.

Theorem.

(a) $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

(b) $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right|\,dx = +\infty.$

Proof. The following proof is outlined on page 397, Miscellaneous Exercise 39, in G. H. Hardy's *A Course of Pure Mathematics*.

(a) For each $n \geq 1$, define

$$u_n = \int_0^{\pi/2} \sin 2nx \cot x\,dx \qquad \text{and} \qquad v_n = \int_0^{\pi/2} \frac{\sin 2nx}{x}\,dx,$$

and consider the difference $u_{n+1} - u_n$,

$$\begin{aligned}
u_{n+1} - u_n &= \int_0^{\pi/2} [\sin 2(n+1)x - \sin 2nx] \cot x\,dx \\
&= 2\int_0^{\pi/2} \sin x \cos(2n+1)x \cot x\,dx \\
&= 2\int_0^{\pi/2} \cos x \cos(2n+1)x\,dx \\
&= \int_0^{\pi/2} [\cos(2n+2)x + \cos 2nx]\ dx \\
&= \frac{1}{2(n+1)} \sin 2(n+1)x \Big|_0^{\pi/2} + \frac{1}{2n} \sin 2nx \Big|_0^{\pi/2} \\
&= 0.
\end{aligned}$$

Therefore, $u_{n+1} - u_n = 0$ for all $n \geq 1$. Thus, u_n is constant for $n \geq 1$, and

$$\begin{aligned} u_1 &= \int_0^{\pi/2} \sin 2x \cot x \, dx = 2 \int_0^{\pi/2} \cos^2 x \, dx \\ &= \int_0^{\pi/2} (1 + \cos 2x) \, dx = \frac{\pi}{2} + \left. \frac{\sin 2x}{2} \right|_0^{\pi/2} \\ &= \frac{\pi}{2}, \end{aligned}$$

and therefore $u_n = u_1 = \pi/2$ for all $n \geq 1$.

Making the substitution $t = 2nx$ in the integral $\int_0^{\pi/2} (\sin 2nx/x) \, dx$, then

$$v_n = \int_0^{\pi/2} \frac{\sin 2nx}{x} \, dx = \int_0^{n\pi} \frac{\sin t}{t} \, dt, \tag{1}$$

and integrating by parts, we have

$$\begin{aligned} u_n - v_n &= \int_0^{\pi/2} \left(\cot x - \frac{1}{x} \right) \sin 2nx \, dx \\ &= - \frac{\cos 2nx}{2n} \left(\cot x - \frac{1}{x} \right) \Bigg|_0^{\pi/2} \\ &\qquad - \frac{1}{2n} \int_0^{\pi/2} \left(\csc^2 x - \frac{1}{x^2} \right) \cos 2nx \, dx, \end{aligned}$$

and therefore

$$\begin{aligned} u_n - v_n = \frac{(-1)^n}{n\pi} &+ \lim_{p \to 0^+} \frac{\cos 2np}{2n} \left(\cot p - \frac{1}{p} \right) \\ &- \frac{1}{2n} \int_0^{\pi/2} \left(\csc^2 x - \frac{1}{x^2} \right) \cos 2nx \, dx. \end{aligned} \tag{2}$$

Using L'Hospital's rule, we have

$$\lim_{p \to 0^+} \left(\cot p - \frac{1}{p} \right) = 0, \quad \text{and} \quad \lim_{x \to 0^+} \left(\csc^2 x - \frac{1}{x^2} \right) = \frac{1}{3}.$$

Therefore, we may redefine the integrand on the right-hand side of (2) to be continuous on the interval $[0, \pi/2]$, and hence bounded there. Thus, there exists a constant $M > 0$ such that

$$\begin{aligned} \left| \int_0^{\pi/2} \left(\csc^2 x - \frac{1}{x^2} \right) \cos 2nx \, dx \right| &\leq \int_0^{\pi/2} \left| \left(\csc^2 x - \frac{1}{x^2} \right) \cos 2nx \right| dx \\ &\leq \frac{\pi M}{2}. \end{aligned}$$

Letting $n \to \infty$ in (1) and (2), we obtain

$$\lim_{n \to \infty} v_n = \lim_{n \to \infty} (v_n - u_n + u_n) = \lim_{n \to \infty} (v_n - u_n) + \lim_{n \to \infty} u_n,$$

so that

$$\int_0^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}.$$

(b) For each positive integer n, we have

$$\int_0^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx = \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx$$

$$= \sum_{k=0}^{n} \int_0^{\pi} \left| \frac{\sin(t+k\pi)}{t+k\pi} \right| dt = \sum_{k=0}^{n} \int_0^{\pi} \left| \frac{(-1)^k \sin t}{t+k\pi} \right| dt$$

$$= \sum_{k=0}^{n} \int_0^{\pi} \frac{|\sin t|}{t+k\pi}\, dt = \sum_{k=0}^{n} \int_0^{\pi} \frac{\sin t}{t+k\pi}\, dt,$$

where the last equality follows since $\sin t \geq 0$ for $0 \leq t \leq \pi$. For $0 < t < \pi$, we have $1/(t+k\pi) > 1/[\pi(k+1)]$, so that

$$\int_0^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx > \sum_{k=0}^{n} \frac{1}{\pi(k+1)} \int_0^{\pi} \sin t\, dt = \frac{2}{\pi} \sum_{k=0}^{n} \frac{1}{k+1}.$$

Now, for $x \geq 1$ we have

$$\log x = \int_1^x \frac{1}{t}\, dt \leq \int_1^x 1\, dt = x - 1,$$

and replacing x by $(k+2)/(k+1)$, we have

$$\log\left(\frac{k+2}{k+1}\right) \leq \frac{1}{k+1}$$

for all $k \geq 0$. Therefore,

$$\sum_{k=0}^{n} \frac{1}{k+1} \geq \sum_{k=0}^{n} (\log(k+2) - \log(k+1)) = \log(n+2),$$

so

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k+1} \geq \lim_{n\to\infty} \log(n+2) = +\infty,$$

and this implies that

$$\int_0^{\infty} \left| \frac{\sin x}{x} \right| dx = +\infty.$$

□

It is instructive to compare the real variable proof of part (a) of this theorem with a proof using Cauchy's residue theorem.