Math 309 - Spring-Summer 2017

Problem Set # 9

Completion Date: July 7, 2017

Question 1. Show in two ways that the sequence

$$z_n = -2 + i\,\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \dots)$$

converges to -2.

Question 2.

Let r_n denote the moduli and Θ_n the principal values of the arguments of the complex numbers

$$z_n = -2 + \frac{i(-1)^n}{n^2}$$

for $n \geq 1$. Show that the sequence r_n $(n = 1, 2, \dots)$ converges but that the sequence Θ_n $(n = 1, 2, \dots)$ does not.

Question 3.

(a) Show that if the sequence $\{z_n\}_{n\geq 1}$ converges, then $(z_n - z_{n-1}) \to 0$ as $n \to \infty$.

(b) Let $z_0 \neq 0$. Show that the sequence $\{(z/z_0)^n\}_{n\geq 1}$ diverges if $|z| = |z_0|$ and $z \neq z_0$.

Hint: For $|z| = |z_0|$, show first that

$$\left|\left(\frac{z}{z_0}\right)^n - \left(\frac{z}{z_0}\right)^{n-1}\right| = \left|\frac{z}{z_0} - 1\right| > 0,$$

and use the result of part (a).

Question 4.

Show that

$$\text{if} \qquad \lim_{n\to\infty} z_n = z, \qquad \text{then} \qquad \lim_{n\to\infty} |z_n| = |z|.$$

Question 5.

Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

Question 6.

Obtain the Taylor series

$$e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

for the function $f(z) = e^z$ by

(a) using $f^{(n)}(1) \quad (n = 0, 1, 2 \ldots)$;

(b) writing $e^z = e^{z-1}e$.

Question 7.

Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}.$$

Ans: $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \qquad (|z| < \sqrt{3})$.

Question 8.

With the aid of the identity

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$

expand $\cos z$ into a Taylor series about the point $z_0 = \pi/2$.

Question 9.

What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

Question 10.

Show that when $z \neq 0$,

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots.$$

Question 11.

Represent the function

$$f(z) = \frac{z+1}{z-1}$$

(a) by its Maclaurin series, and state where the representation is valid;

(b) by its Laurent series in the domain $1 < |z| < \infty$.

Ans: (a) $-1 - 2\sum_{n=1}^{\infty} z^n \quad (|z| < 1)$; (b) $1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n}$.

Question 12.

Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Question 13.

Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Ans: $\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z} \quad (0 < |z| < 1);\qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{z^{2n+1}} \quad (1 < |z| < \infty).$