Math 309 - Spring-Summer 2017

Problem Set # 8

Completion Date: Friday June 30, 2017

Question 1.

By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

$$\text{(a)} \int_i^{i/2} e^{\pi z}\, dz; \qquad \text{(b)} \int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\, dz; \qquad \text{(c)} \int_1^3 (z-2)^3\, dz.$$

Ans: (a) $(1+i)/\pi$; (b) $e + (1/e)$; (c) 0.

Question 2.

Show that

$$\int_{-1}^{1} z^i\, dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where z^i denotes the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0,\ -\pi < \operatorname{Arg} z < \pi)$$

and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis.

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i \log z) \quad \left(|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

Question 3.

Apply the Cauchy-Goursat theorem to show that $\displaystyle\int_C f(z)\, dz = 0$ when the contour C is the circle $|z| = 1$, in either direction, and when $f(z) = \dfrac{z^2}{z-3}$.

Question 4.

Apply the Cauchy-Goursat theorem to show that $\displaystyle\int_C f(z)\, dz = 0$ when the contour C is the circle $|z| = 1$, in either direction, and when $f(z) = \dfrac{1}{z^2+2z+2}$.

Question 5.

Apply the Cauchy-Goursat theorem to show that $\int_C f(z)\,dz = 0$ when the contour C is the circle $|z| = 1$, in either direction, and when $f(z) = \text{Log}(z+2)$.

Question 6.

Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral

$$\int_C \frac{e^{-z}}{z - (\pi i/2)}\,dz.$$

Ans: 2π.

Question 7.

Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz.$$

Ans: $\pi i/4$.

Question 8.

Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz \quad (-2 < x_0 < 2).$$

Ans: $i\pi \sec^2(x_0/2)$.

Question 9.

Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2+4}$; (b) $g(z) = \dfrac{1}{(z^2+4)^2}$.

Ans: (a) $\pi/2$; (b) $\pi/16$.

Question 10.

Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\,dz \quad (|w| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

Question 11.

Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that, for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$