Math 309 - Spring-Summer 2017

Problem Set # 7

Completion Date: Friday June 23, 2017

Question 1.

Evaluate the following integrals:

$$\text{(a)} \int_1^2 \left(\frac{1}{t} - i\right)^2 dt; \qquad \text{(b)} \int_0^{\pi/6} e^{i2t}\, dt; \qquad \text{(c)} \int_0^{\infty} e^{-zt}\, dt \quad (\text{Re } z > 0).$$

Ans: (a) $-\frac{1}{2} - i \ln 4$; (b) $\frac{\sqrt{3}}{4} + \frac{i}{4}$; (c) $\frac{1}{z}$.

Question 2.

Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta}\, e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{when} \quad m \neq n, \\ 2\pi & \text{when} \quad m = n. \end{cases}$$

Question 3.

According to the definition of integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^x \cos x\, dx + i \int_0^{\pi} e^x \sin x\, dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

Ans: $-(1 + e^{\pi})/2$, $(1 + e^{\pi})/2$.

Question 4.

Use parametric representations for the contour C, or legs of C, to evaluate

$$\int_C f(z)\, dz$$

when $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

(a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$;

(b) the segment $0 \leq x \leq 2$ of the real axis.

Ans: (a) 0; (b) 0.

Question 5.

Use parametric representations for the contour C, or legs of C, to evaluate

$$\int_C f(z)\, dz$$

when $f(z)$ is defined by the equations

$$f(z) = \begin{cases} 1 & \text{when} \quad y < 0, \\ 4y & \text{when} \quad y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Ans: $2 + 3i$.

Question 6.

Use parametric representations for the contour C, or legs of C, to evaluate

$$\int_C f(z)\, dz$$

when $f(z)$ is the branch

$$z^{-1+i} = \exp\left[(-1+i)\log z\right] \qquad (|z| > 0,\ 0 < \arg z < 2\pi)$$

of the indicated power function, and C is the positively oriented unit circle $|z| = 1$.

Ans: $i\left(1 - e^{-2\pi}\right)$.

Question 7.

Let C_0 denote the circle of radius R centered at z_0, $|z - z_0| = R$, taken counterclockwise. Use the parametric representation $z = z_0 + Re^{i\theta}$ $(-\pi \le \theta \le \pi)$ for C_0 to derive the following integration formulas:

(a) $\displaystyle\int_{C_0} \frac{dz}{z - z_0} = 2\pi i;$

(b) $\displaystyle\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \quad (n = \pm 1, \pm 2, \ldots).$

Question 8.

Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.