Math 309 - Spring-Summer 2017

Problem Set # 4

Completion Date: Friday June 2, 2017

Question 1.

Verify that the function

$$f(z) = 3x + y + i\,(3y - x)$$

is entire.

Question 2.

Verify that the function

$$f(z) = e^{-y}\sin x - i\,e^{-y}\cos x$$

is entire.

Question 3.

For the function

$$f(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)},$$

determine the singular points of the function and state why the function is analytic everywhere except at those points.

Ans: $z = -2,\ -1 \pm i$.

Question 4.

Verify that the function

$$g(z) = \ln r + i\,\theta \quad (r > 0,\ 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = \dfrac{1}{z}$. Then show that the composite function $G(z) = g(z^2+1)$ is analytic in the quadrant $x > 0,\ y > 0$, with derivative

$$G'(z) = \frac{2z}{z^2+1}.$$

Suggestion: Observe that $\mathrm{Im}(z^2+1) > 0$ when $x > 0,\ y > 0$.

Question 5.

Let $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant throughout D if $|f(z)|$ is constant throughout D.

Suggestion: Observe that

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \text{if} \quad |f(z)| = c\ (c \neq 0).$$

Question 6.

Show that the function

$$u(x, y) = \sinh x \sin y$$

is harmonic in some domain and find a harmonic conjugate $v(x, y)$.

Ans: $v(x, y) = -\cosh x \cos y$.

Question 7.

Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0,\ 0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation. Then use the Cauchy-Riemann equations in polar form, to derive the harmonic conjugate $v(r, \theta) = \theta$.

Question 8.

Show that

(a) $\exp(2 \pm 3\pi\, i) = -e^2$; (b) $\exp\left(\dfrac{2 + \pi\, i}{4}\right) = \sqrt{\dfrac{e}{2}}(1 + i)$; (c) $\exp(z + \pi\, i) = -\exp z$.

Question 9.

Use the Cauchy-Riemann equations to show that the function

$$f(z) = \exp \overline{z}$$

is not analytic anywhere.

Question 10.

Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp(2z + i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}.$$

Question 11.

Show that $|\exp(z^2)| \le \exp(|z|^2)$.

Question 12.

Find all values of z such that

(a) $e^z = -2$; (b) $e^z = 1 + \sqrt{3}\, i$; (c) $\exp(2z - 1) = 1$.

Ans:

(a) $z = \ln 2 + (2n + 1)\pi\, i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

(b) $z = \ln 2 + \left(2n + \dfrac{1}{3}\right)\pi\, i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

(c) $z = \dfrac{1}{2} + n\pi\, i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

Question 13. We showed in class that for the inversion mapping $f(z) = 1/z$, $z \neq 0$, the real and imaginary parts of $f(z)$ are

$$u(x, y) = \frac{x}{x^2 + y^2} \qquad \text{and} \qquad v(x, y) = \frac{-y}{x^2 + y^2}.$$

Show that the level curves of $u(x, y)$ are a family of circles passing through the origin with center on the real axis; while the level curves of $v(x, y)$ are a family of circles passing through the origin with center on the imaginary axis.