Math 309 - Spring-Summer 2017

Problem Set # 3

Completion Date: Friday May 26, 2017

Question 1.

Show that $\lim\limits_{z\to\infty} \dfrac{4z^2}{(z-1)^2} = 4$.

Question 2.

Show that a set is unbounded if and only if every neighborhood of the point at infinity contains at least one point in S.

Question 3.

Find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$;

(b) $f(z) = \left(1 - 4z^2\right)^3$;

(c) $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq -1/2)$;

(d) $f(z) = \dfrac{\left(1+z^2\right)^4}{z^2}$ $(z \neq 0)$.

Question 4.

Apply the definition of the derivative to give a direct proof that

$$f'(z) = -\frac{1}{z^2} \qquad \text{when} \qquad f(z) = \frac{1}{z} \quad (z \neq 0).$$

Question 5.

Show that $f'(z)$ does not exist at any point z when $f(z) = \operatorname{Im} z$.

Question 6.

Show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$;

(b) $f(z) = z - \overline{z}$;

(c) $f(z) = 2x + i\,xy^2$;

(d) $f(z) = e^x e^{-i\,y}$.

Question 7.

Determine where $f'(z)$ exists and find its value when

(a) $f(z) = \dfrac{1}{z}$; (b) $f(z) = x^2 + i\,y^2$; (c) $f(z) = z\,\mathrm{Im}\; z$;

Ans: (a) $f'(z) = -\dfrac{1}{z^2}$ $(z \neq 0)$; (b) $f'(x + i\,x) = 2x$; (c) $f'(0) = 0$.

Question 8.

Show that the function

$$f(z) = \sqrt{r}e^{i\,\theta/2} \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

is differentiable in the indicacted domain of definition, and then find $f'(z)$.

Ans: $f'(z) = \dfrac{1}{2f(z)}$.

Question 9.

Show that when $f(z) = x^3 + i\,(1 - y)^3$, it is legitimate to write

$$f'(z) = u_x + i\,v_x = 3x^2$$

only when $z = i$.

Question 10.

(a) Recall that if $z = x + i\,y$ then

$$x = \frac{z + \overline{z}}{2} \qquad \text{and } y = \frac{z - \overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y}\right).$$

(b) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary parts of a function $f(z) = u(x, y) + i\,v(x, y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[(u_x - v_y) + i\,(v_x + u_y)\right] = 0.$$

Thus derive the *complex form* $\dfrac{\partial f}{\partial \overline{z}} = 0$ *of the Cauchy-Riemann equations.*