Math 309 - Spring-Summer 2017

Problem Set # 2

Completion Date: Friday May 19, 2017

Question 1.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

(a) $(-16)^{1/4}$; (b) $\left(-8-8\sqrt{3}\,i\right)^{1/4}$.

Ans: (a) $\pm\sqrt{2}\,(1+i)$, $\pm\sqrt{2}\,(1-i)$; (b) $\pm(\sqrt{3}-i)$, $\pm(1+\sqrt{3}\,i)$.

Question 2.

In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(a) $(-1)^{1/3}$; (b) $8^{1/6}$.

Ans: (b) $\pm\sqrt{2}$, $\pm\dfrac{1+\sqrt{3}\,i}{\sqrt{2}}$, $\pm\dfrac{1-\sqrt{3}\,i}{\sqrt{2}}$.

Question 3.

Sketch the following sets and determine which are domains:

(a) $|z-2+i|\le 1$; (b) $|2z+3|>4$; (c) $\text{Im } z>1$;
(d) $\text{Im } z=1$; (e) $0\le \arg z\le \pi/4$ $(z\neq 0)$; (f) $|z-4|\ge|z|$.

Ans: (b), (c) are domains.

Question 4.

In each case, sketch the closure of the set:

(a) $-\pi<\arg z<\pi$ $(z\neq 0)$; (b) $|\text{Re } z|<|z|$;
(c) $\text{Re}\left(\dfrac{1}{z}\right)\le\dfrac{1}{2}$; (d) $\text{Re}\left(z^2\right)>0$.

Question 5.

Write the function $f(z)=z^3+z+1$ in the form $f(z)=u(x,y)+i\,v(x,y)$.

Ans: $(x^3-3xy^2+x+1)+i\,(3x^2y-y^3+y)$.

Question 6.

Suppose that $f(z) = x^2 - y^2 - 2y + i\,(2x - 2xy)$, where $z = x + i\,y$. Use the expressions

$$x = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad y = \frac{z - \overline{z}}{2i}$$

to write $f(z)$ in terms of z and simplify the result.

Ans: $\overline{z}^2 + 2i\,z$.

Question 7.

Find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1,\ u = 2,\ v = 1,$ and $v = 2$.

Question 8.

Sketch the region onto which the sector $r \leq 1,\ 0 \leq \theta \leq \pi/4$ is mapped by the transformation

(a) $w = z^2$; (b) $w = z^3$; (c) $w = z^4$.

Question 9.

(a) Describe and sketch the set

$$\mathcal{D} = \left\{\, z \in \mathbb{C} \mid 2\,\mathrm{Re}(z^2) = |z|^2 \,\right\}.$$

(b) Describe and sketch the set

$$\mathcal{D} = \left\{\, z \in \mathbb{C} \mid \mathrm{Im}\left(\frac{1}{z}\right) > 1 \,\right\}.$$

Question 10.

(a) Given a positive integer $n > 2$, find all complex numbers $z \in \mathbb{C}$ satisfying

$$\overline{z} = z^{n-1}.$$

(b) Let ω_n be the primitive n^{th} root of unity given by $e^{\frac{2\pi i}{n}}$, $n \geq 2$, calculate

$$1 + 2\,\omega_n + 3\,\omega_n^2 + \cdots + n\,\omega_n^{n-1}.$$