Math 309 - Spring-Summer 2017

Problem Set # 12

Completion Date: Friday August 4, 2017

Question 1.

Use residues to evaluate the improper integral

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

Ans: $\pi/4$.

Question 2.

Use residues to evaluate the improper integral

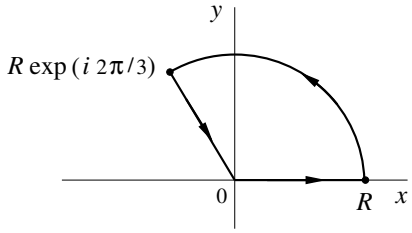
$$\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}.$$

Ans: $\pi/200$.

Question 3.

Use residues and the contour shown in the figure, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$



Question 4.

Use residues to evaluate the improper integral

$$\int_{-\infty}^\infty \frac{x\sin ax}{x^4+4}\,dx \quad (a>0).$$

Ans: $\dfrac{\pi}{2}e^{-a}\sin a$.

Question 5.

Use residues to find the Cauchy principal value of the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2 + 4x + 5}.$$

Ans: $-\dfrac{\pi}{e} \sin 2.$

Question 6.

Evaluate the improper integral

$$\int_0^{\infty} \frac{x^a}{(x^2+1)^2} \, dx, \quad \text{where} \quad -1 < a < 3 \quad \text{and} \quad x^a = \exp(a \ln x).$$

Ans: $\dfrac{(1-a)\pi}{4\cos(a\pi/2)}.$

Question 7.

Use residues to evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}.$$

Ans: $\sqrt{2}\,\pi.$

Question 8.

Use residues to evaluate the definite integral

$$\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2} \quad (-1 < a < 1).$$

Ans: $\dfrac{a^2\pi}{1-a^2}.$

Question 9.

Use residues to evaluate the definite integral

$$\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} \quad (a > 1).$$

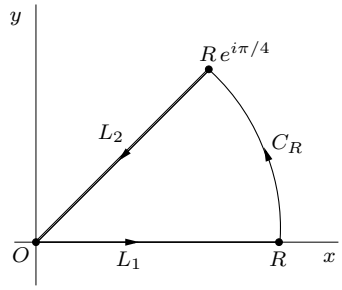
Ans: $\dfrac{a\pi}{\left(\sqrt{a^2-1}\right)^3}.$

Question 10.

Evaluate the *Fresnel integrals*:

$$\int_0^\infty \cos(x^2)\,dx \qquad \text{and} \qquad \int_0^\infty \sin(x^2)\,dx$$

by integrating $f(z) = e^{iz^2}$ around the boundary $\mathcal{C}$ of the sector $0 \le r \le R$, and $0 \le \theta \le \pi/4$ shown below and using the Cauch y-Goursat theorem.



Ans: $\sqrt{\dfrac{\pi}{8}}$.