Math 309 - Spring-Summer 2017

Problem Set # 11

Completion Date: Friday July 28, 2017

Question 1.

Find the residue at $z = 0$ of the function

(a) $\dfrac{1}{z + z^2}$; (b) $z \cos\left(\dfrac{1}{z}\right)$;

(c) $\dfrac{z - \sin z}{z}$; (d) $\dfrac{\cot z}{z^4}$;

(e) $\dfrac{\sinh z}{z^4(1 - z^2)}$.

Ans: (a) 1; (b) $-1/2$; (c) 0; (d) $-1/45$; (e) $7/6$.

Question 2.

Use Cauchy's residue theorem to evaluate the integral of

$$\frac{\exp(-z)}{z^2}$$

around the circle $|z| = 3$ in the positive sense.

Ans: $-2\pi i$.

Question 3.

Use Cauchy's residue theorem to evaluate the integral of

$$z^2 \exp\left(\frac{1}{z}\right)$$

around the circle $|z| = 3$ in the positive sense.

Ans: $\pi i/3$.

Question 4.

Use Cauchy's residue theorem to evaluate the integral of

$$\frac{z + 1}{z^2 - 2z}$$

around the circle $|z| = 3$ in the positive sense.

Ans: $2\pi i$.

Question 5.

In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $z \exp\left(\dfrac{1}{z}\right)$; (b) $\dfrac{z^2}{1+z}$; (c) $\dfrac{\sin z}{z}$; (d) $\dfrac{\cos z}{z}$; (e) $\dfrac{1}{(2-z)^3}$.

Question 6.

Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; (b) $\dfrac{1-\exp(2z)}{z^4}$; (c) $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans: (a) $m=1,\ B=-1/2$; (b) $m=3,\ B=-4/3$; (c) $m=2,\ B=2e^2$.

Question 7.

In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z^2+2}{z-1}$; (b) $\left(\dfrac{z}{2z+1}\right)^3$; (c) $\dfrac{\exp(z)}{z^2+\pi^2}$.

Ans: (a) $m=1,\ B=3$; (b) $m=3,\ B=-3/16$; (c) $m=1,\ B=\pm i/2\pi$.

Question 8.

Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz,$$

taken counterclockwise around the circle

(a) $|z-2|=2$; (b) $|z|=4$.

Ans: (a) πi; (b) $6\pi i$.

Question 9.

Show that

(a) $\underset{z=\pi i}{\mathrm{Res}} \dfrac{z-\sinh z}{z^2 \sinh z} = \dfrac{i}{\pi}$;

(b) $\underset{z=\pi i}{\mathrm{Res}} \dfrac{\exp(zt)}{\sinh z} + \underset{z=-\pi i}{\mathrm{Res}} \dfrac{\exp(zt)}{\sinh z} = -2\cos \pi t$.