Math 309 - Spring-Summer 2017

Problem Set # 1

Completion Date: Friday May 12, 2017

Question 1.

Show that

(a) $\text{Re}\,(iz) = -\text{Im}\ z$; (b) $\text{Im}\,(iz) = \text{Re}\ z$.

Question 2.

Reduce the quantity $\dfrac{5i}{(1-i)(2-i)(3-i)}$ to a real number.

Question 3.

Reduce the quantity $(1-i)^4$ to a real number.

Question 4.

Verify that $\sqrt{2}\,|z| \geq |\text{Re}\ z| + |\text{Im}\ z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Question 5.

In each case, sketch the set of points determined by the given condition:

(a) $|z - 1 + i| = 1$; (b) $|z + i| \leq 3$; (c) $|z - 4i| \geq 4$.

Question 6.

Use the properties of conjugates and modulii established in class to show that

(a) $\overline{\overline{z} + 3i} = z - 3i$ (b) $\overline{iz} = -i\,\overline{z}$;

(c) $\overline{(2+i)^2} = 3 - 4i$; (d) $\left|(2\overline{z} + 5)(\sqrt{2} - i)\right| = \sqrt{3}\,|2z + 5|$.

Question 7.

Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{\big|\,|z_3| - |z_4|\,\big|}.$$

Question 8.

Find the principal argument Arg z when $z = \dfrac{i}{-2-2i}$.

Ans. $-\dfrac{3\pi}{4}$.

Question 9.

Using the fact that the modulus $\left|e^{i\theta} - 1\right|$ is the distance between the points $e^{i\theta}$ and 1, give a geometric argument to find a value of θ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $\left|e^{i\theta} - 1\right| = 2$.

Ans. π.

Question 10.

Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity*:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[(2n+1)\theta/2\right]}{2\,\sin\left(\theta/2\right)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.