Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers
Solutions to Practice Problems for Final Examination
Completion Date: Tuesday August 15, 2017

Question 1.

(a) Using a purely geometric argument, show that

$$|z-1| \le \big||z|-1\big| + |z|\,|\arg(z)|$$

for all $z \in \mathbb{C}$.

(b) Let ω_n be the primitive n^{th} root of unity given by $e^{\frac{2\pi i}{n}}$, $n \ge 2$. Calculate

$$1 + 4\omega_n + 9\omega_n^2 + \cdots + n^2\omega_n^{n-1}.$$

Solution:

(a) From the triangle inequality, we have

$$|z-1| \le \big||z|-1\big| + \big|z-|z|\big|,$$

but $\big|z-|z|\big|$ is the length of the chord joining the point z and $|z|$, while $|z|\cdot\arg(z)$ is the length of the circular arc from z to $|z|$ and

$$\big|z-|z|\big| \le |z|\cdot\arg(z),$$

so that

$$|z-1| \le \big||z|-1\big| + |z|\cdot\arg(z).$$

(b) Since $n \ge 1$, then $w_n = e^{\frac{2\pi i}{n}} \ne 1$, and since

$$0 = 1 - w_n^n = (1-w_n)\left(1 + w_n + w_n^2 + \cdots w_n^{n-1}\right)$$

then

$$1 + w_n + w_n^2 + \cdots w_n^{n-1} = 0.$$

Letting $T_n = 1 + 2w_n + 3w_n^2 + \cdots + nw_n^{n-1}$, we have

$$(1-w_n)T_n = 1 + w_n + w_n^2 + \cdots + w_n^{n-1} - nw_n^n = -n,$$

so that

$$1 + 2w_n + 3w_n^2 + \cdots + nw_n^{n-1} = -\frac{n}{1-w_n}.$$

Now let $S_n = 1 + 4w_n + 9w_n^2 + \cdots + n^2w_n^{n-1}$, then

$$\begin{aligned}(1-w_n)S_n &= 1 + 3w_n + 5w_n^2 + 7w_n^3 + \cdots + (2n-1)w_n^{n-1} - n^2\\ &= 2 + 4w_n + 6w_n^2 + 8w_n^3 + \cdots + 2nw_n^{n-1} - n^2,\end{aligned}$$

since $1 + w_n + w_n^2 + \cdots w_n^{n-1} = 0$. Therefore,

$$(1-w_n)S_n = 2\left(1 + 2w_n + 3w_n^2 + \cdots + nw_n^{n-1}\right) - n^2 = -\frac{2n}{1-w_n} - n^2,$$

so that

$$S_n = -\frac{2n}{(1-w_n)^2} - \frac{n^2}{1-w_n}.$$

Question 2.

Let $\alpha,\, \beta \in \mathbb{C}$.

(a) Show that $\quad |\alpha + \beta\,|^2 = \left|1 + \alpha\overline{\beta}\,\right|^2 - \left(1 - |\alpha|^2\right)\left(1 - |\beta|^2\right)$.

(b) Show that if $|\alpha| \le 1$ and $|\beta| \le 1$, then $\quad |\alpha + \beta\,| \le \left|1 + \alpha\overline{\beta}\,\right|$.

(c) When does equality hold in part (b)?

Solution:

(a) For $\alpha,\, \beta \in \mathbb{C}$ we have

$$|\alpha + \beta|^2 = (\alpha + \beta)(\overline{\alpha} + \overline{\beta}) = |\alpha|^2 + \alpha\overline{\beta} + \overline{\alpha}\beta + |\beta|^2,$$

also,

$$|1 + \alpha\overline{\beta}|^2 = (1 + \alpha\overline{\beta})(1 + \overline{\alpha}\beta) = 1 + \alpha\overline{\beta} + \overline{\alpha}\beta + |\alpha|^2\,|\beta|^2,$$

so that

$$|\alpha + \beta|^2 = |\alpha|^2 + |1 + \alpha\overline{\beta}|^2 - 1 - |\alpha|^2\,|\beta|^2 + |\beta|^2,$$

that is,

$$|\alpha + \beta|^2 = |1 + \alpha\overline{\beta}|^2 - (1 - |\alpha|^2)(1 - |\beta|^2). \tag{$*$}$$

(b) If $|\alpha| \le 1$ and $|\beta| \le 1$, then

$$(1 - |\alpha|^2)(1 - |\beta|^2) \ge 0,$$

and therefore

$$|\alpha + \beta|^2 \le |1 + \alpha\overline{\beta}|^2,$$

taking the nonnegative square root of both sides of this inequality we have

$$|\alpha + \beta| \le |1 + \alpha\overline{\beta}|.$$

(c) From $(*)$, equality holds if and only if either $|\alpha| = 1$ or $|\beta| = 1$.

Question 3.

Let $\alpha = \frac{m}{n}$, where m and n are positive integers, and let $z^\alpha = e^{\alpha \log z}$ denote the multiple-valued α^{th} power function for $z \ne 0$.

(a) Show that the principal value of $\left(z^{\frac{1}{n}}\right)^m$ always gives the principal value of z^α.

(b) Show that $(z^m)^{\frac{1}{n}}$ may *not* give the correct values of z^α by calculating the principal values of

$$\left(z^2\right)^{\frac{1}{2}} \qquad\qquad \left(z^{\frac{1}{2}}\right)^2 \qquad\qquad z^1$$

for $z = -1 + i$.

(c) What, if anything, is wrong with the following?

$$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1} = i \cdot i = -1$$

Solution: Let $\alpha = \frac{m}{n}$ where m and n are positive integers, if $z \neq 0$, then the principal value of z^α is

$$z^\alpha = e^{\alpha \operatorname{Log} z} = e^{\alpha[\ln|z| + i\operatorname{Arg}(z)]},$$

that is,

$$z^\alpha = |z|^\alpha \cdot e^{i\alpha \operatorname{Arg}(z)}$$

where $|z| > 0$ and $-\pi < \operatorname{Arg}(z) \leq \pi$.

(a) The principal value of $z^{\frac{1}{n}}$ is

$$z^{\frac{1}{n}} = \sqrt[n]{|z|} \cdot e^{\frac{i\operatorname{Arg}(z)}{n}},$$

and so the principal value of $\left(z^{\frac{1}{n}}\right)^m$ is

$$\left(z^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{|z|}\right)^m \cdot e^{i\frac{m}{n}\operatorname{Arg}(z)} = |z|^\alpha \cdot e^{i\alpha \operatorname{Arg}(z)}$$

where $|z| > 0$ and $-\pi < \operatorname{Arg}(z) \leq \pi$, which is exactly the same as the principal value of $z^\alpha = e^{\alpha \operatorname{Log} z}$.

(b) If $z = -1 + i$, then $z^2 = (-1+i)^2 = 1 - 2i - 1 = -2i$, and the principal value of $\left(z^2\right)^{\frac{1}{2}}$ is

$$\left(z^2\right)^{\frac{1}{2}} = \sqrt{2} \cdot e^{-\frac{i\pi}{4}} = 1 - i.$$

If $z = -1 + i$, the principle value of $z^{\frac{1}{2}}$ is

$$z^{\frac{1}{2}} = \sqrt[4]{2} \cdot e^{\frac{i3\pi}{8}},$$

so that

$$\left(z^{\frac{1}{2}}\right)^2 = \sqrt{2} \cdot e^{\frac{i3\pi}{4}} = -1 + i,$$

which is the same as $z^1 = -1 + i$. Therefore, $(z^2)^{\frac{1}{2}} \neq \left(z^{\frac{1}{2}}\right)^2$.

(c) From the above, it is not true in general that $(z^2)^{\frac{1}{2}} = \left(z^{\frac{1}{2}}\right)^2$, in particular,

$$\sqrt{1} = \sqrt{(-1)(-1)} \neq \sqrt{-1} \cdot \sqrt{-1} = i^2 = -1.$$

Question 4.

Show that for each positive integer $n \geq 1$, and for each real number α,

$$(1 + \cos\alpha + i\,\sin\alpha)^n = 2^n \cos^n \frac{\alpha}{2}\left(\cos\frac{n\alpha}{2} + i\,\sin\frac{n\alpha}{2}\right).$$

Solution: If $n \geq 1$ and α is a real number, then

$$\begin{aligned}
(1 + \cos\alpha + i\sin\alpha)^n &= \left(2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)^n \\
&= 2^n \cos^n\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)^n \\
&= 2^n \cos^n\frac{\alpha}{2} \cdot \left(e^{\frac{i\alpha}{2}}\right)^n \\
&= 2^n \cos^n\frac{\alpha}{2} \cdot e^{\frac{in\alpha}{2}} \\
&= 2^n \cos^n\frac{\alpha}{2}\left(\cos\frac{n\alpha}{2} + i\sin\frac{n\alpha}{2}\right),
\end{aligned}$$

so that

$$(1 + \cos\alpha + i\,\sin\alpha)^n = 2^n \cos^n \frac{\alpha}{2}\left(\cos\frac{n\alpha}{2} + i\,\sin\frac{n\alpha}{2}\right).$$

Question 5.

Show that

$$\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdot\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}=\frac{n}{2^{n-1}}$$

for $n=2,3,\ldots$.

SOLUTION: For $n\geq 2$, the roots of the equation $z^n-1=0$ are

$$z_0=1,\ z_1=e^{\frac{2\pi i}{n}},\ z_2=e^{\frac{4\pi i}{n}},\ \ldots,\ z_{n-1}=e^{\frac{2(n-1)\pi i}{n}},$$

so that the expression z^n-1 factors as

$$z^n-1=(z-1)\big(z-e^{\frac{2\pi i}{n}}\big)\big(z-e^{\frac{4\pi i}{n}}\big)\cdots\big(z-e^{\frac{2(n-1)\pi i}{n}}\big).$$

Now divide by $z-1$ and let $z\to 1$ to get

$$\frac{d}{dz}(z^n)\bigg|_{z=1}=n,$$

so that

$$n=\big(1-e^{\frac{2\pi i}{n}}\big)\big(1-e^{\frac{4\pi i}{n}}\big)\cdots\big(1-e^{\frac{2(n-1)\pi i}{n}}\big). \tag{$*$}$$

Taking complex conjugates, we get

$$n=\overline{n}=\big(1-e^{-\frac{2\pi i}{n}}\big)\big(1-e^{-\frac{4\pi i}{n}}\big)\cdots\big(1-e^{-\frac{2(n-1)\pi i}{n}}\big). \tag{$**$}$$

Multiplying $(*)$ and $(**)$ we get

$$n^2=2^{n-1}\left(1-\cos\frac{2\pi}{n}\right)\left(1-\cos\frac{4\pi}{n}\right)\cdots\left(1-\cos\frac{2(n-1)\pi}{n}\right),$$

so that

$$n^2=2^{n-1}\cdot 2^{n-1}\cdot\sin^2\frac{\pi}{n}\cdot\sin^2\frac{2\pi}{n}\cdots\sin^2\frac{(n-1)\pi}{n},$$

and taking nonnegative square roots

$$\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdot\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}=\frac{n}{2^{n-1}}$$

for $n=2,3,\ldots$.

Question 6.

Find all solutions to the differential equation

$$f''(z)+\beta^2 f(z)=0$$

where $f(z)$ is an entire function.

Hint: Write

$$f(z)=a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots$$

and solve for the coefficients $a_2,\ a_3,\ \ldots$ in terms of a_0, a_1 and β.

SOLUTION: If $f(z)$ is an entire function which is a solution to the differential equation, then $f(z)$ has a Maclaurin series expansion

$$f(z)=a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n+\cdots$$

which is valid for all $z \in \mathbb{C}$, and since $f(z)$ satisfies the differential equation, then

$$f''(z) + \beta^2 f(z) = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \beta^2 \sum_{n=1}^{\infty} a_n z^n = 0,$$

that is,

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + \beta^2 a_n\right] z^n = 0$$

for all $z \in \mathbb{C}$.

Therefore $(n+2)(n+1)a_{n+2} + \beta^2 a_n = 0$ for all $n \geq 0$, and iterating, we have

$$\begin{aligned} a_2 &= -\frac{\beta^2}{2!} a_0 \\ a_3 &= -\frac{\beta^2}{3!} a_1 \\ a_4 &= \frac{\beta^4}{4!} a_0 \\ a_5 &= \frac{\beta^4}{5!} a_1 \\ &\vdots \end{aligned}$$

and an easy induction argument shows that

$$a_{2n} = \frac{(-1)^n \beta^{2n}}{(2n)!} \qquad \text{and} \qquad a_{2n+1} = \frac{(-1)^n \beta^{2n}}{(2n+1)!}$$

for $n \geq 0$.

If $\beta \neq 0$, the solution is

$$f(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\beta z)^{2n} + \frac{a_1}{\beta} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\beta z)^{2n+1} = a_0 \cos \beta z + \frac{a_1}{\beta} \sin \beta z$$

for $z \in \mathbb{C}$.

If $\beta = 0$, the solution is

$$f(z) = a_0 + a_1 z$$

for $z \in \mathbb{C}$.

Question 7.

Which of the following functions is analytic and/or entire and where? (Give reasons for your answers.)

(a) $f(z) = z^5$

(b) $g(z) = \overline{z}^2$

(c) $h(z) = \dfrac{1}{1 - \cos z}$.

Solution:

(a) $f(z) = z^5$ is entire, since $f'(z) = 5z^4$ exists for all $z \in \mathbb{C}$.

(b) $g(z) = \overline{z}^2 = u(x,y) + iv(x,y)$ where $u(x,y) = x^2 - y^2$, and $v(x,y) = -2xy$. Taking partial derivatives, we have

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2x,$$

and

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = -2y,$$

so the partial derivatives exist and are continuous everywhere. However, the Cauchy-Riemann equations hold if and only if

$$\begin{aligned} 2x &= -2x \\ 2y &= -2y, \end{aligned}$$

that is, if and only if $x = y = 0$. Therefore $g'(z)$ exists if and only if $z = 0$, so that $g(z) = \overline{z}^2$ is nowhere analytic.

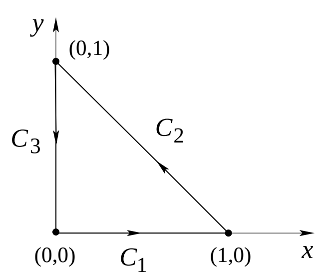
(c) $h(z) = \dfrac{1}{1 - \cos z}$ is analytic at all $z \in \mathbb{C}$ for which $\cos z \neq 1$.

Now $\cos z = 1$ if and only if $\left(e^{iz} + e^{-iz}\right)/2 = 1$, that is, if and only if $e^{2iz} - 2e^{iz} + 1 = 0$, that is, if and only if $\left(e^{iz} - 1\right)^2 = 0$, that is, if and only if $e^{iz} = 1$, that is, if and only if $z = 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$. Therefore, $h(z) = 1/(1 - \cos z)$ is analytic except at the points $z = \pm 2\pi n,\ n \in \mathbb{Z}$.

Question 8.

Compute $\displaystyle\oint_C f(z)\,dz$ where $f(z) = x^2 + i\,y^2$ for $z = x + i\,y$, and C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, and C is traversed in the positive direction.

SOLUTION: We parametrize the contour C below as follows.



On the horizontal line segement C_1 joining the points $(0,0)$ and $(1,0)$, we have

$$z = t,\ 0 \le t \le 1.$$

On the line segment C_2 joining the points $(1,0)$ and $(0,1)$, we have

$$z = (1-t) + it,\ 0 \le t \le 1.$$

On the line segment C_3 joining the points $(0,1)$ and $(0,0)$, we have

$$z = i(1-t),\ 0 \le t \le 1.$$

Therefore,

$$\int_{C_1} f(z)\,dz = \int_0^1 \left[x(t)^2 + iy(t)^2\right]\left[x'(t) + iy'(t)\right]\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$$

$$\int_{C_2} f(z)\,dz = \int_0^1 \left[(1-t)^2 + it^2\right]\left[-1+i\right]\,dt = \frac{1}{3}(1+i)(-1+i) = -\frac{2}{3}$$

$$\int_{C_3} f(z)\,dz = \int_0^1 i(1-t)^2(-i)dt = \frac{1}{3}$$

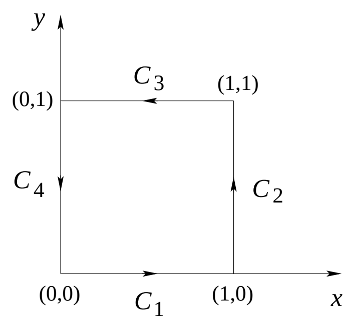
so that

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz = \frac{1}{3} - \frac{2}{3} + \frac{1}{3} = 0.$$

Question 9.

Evaluate the contour integral $\displaystyle\int_C \overline{z}\,dz$

where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ traversed in the counterclockwise direction.



Solution: We write $C = C_1 + C_2 + C_3 + C_4$ as shown in the figure above, then

$$\int_C \overline{z}\,dz = \int_{C_1} \overline{z}\,dz + \int_{C_2} \overline{z}\,dz + \int_{C_3} \overline{z}\,dz + \int_{C_4} \overline{z}\,dz.$$

<u>On C_1</u>: We have $z = t,\ 0 \le t \le 1$, and $\overline{z} = t$, so that

$$\int_{C_1} \overline{z}\,dz = \int_0^1 t\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}.$$

<u>On C_2</u>: We have $z = 1 + it,\ 0 \le t \le 1$, and $\overline{z} = 1 - it$, so that

$$\int_{C_2} \overline{z}\,dz = \int_0^1 (1-it)\,idt = it\Big|_0^1 + \left.\frac{t^2}{2}\right|_0^1 = i + \frac{1}{2}.$$

<u>On C_3</u>: We have $z = t + i,\ 0 \le t \le 1$, and $\overline{z} = t - i$, so that

$$\int_{C_3} \overline{z}\,dz = \int_1^0 (t-i)\,dt = -\int_0^1 (t-i)\,dt = -\left.\frac{t^2}{2}\right|_0^1 + it\Big|_0^1 = -\frac{1}{2} + i.$$

<u>On C_4</u>: We have $z = it,\ 0 \le t \le 1$, and $\overline{z} = -it$, so that

$$\int_{C_4} \overline{z}\,dz = \int_1^0 (-it)\,idt = -\int_0^1 (-it)\,idt = -\left.\frac{t^2}{2}\right|_0^1 = -\frac{1}{2}.$$

Therefore,

$$\int_C \overline{z}\,dz = \frac{1}{2} + i + \frac{1}{2} - \frac{1}{2} + i - \frac{1}{2} = 2i.$$

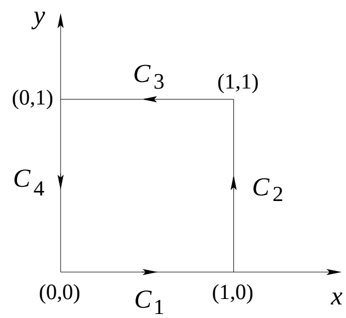
Question 10.

Let C be the boundary of the square with vertices at the points $z = 0, z = 1, z = 1 + i$, $z = i$ and with counterclockwise orientation. Evaluate

$$\oint_C \overline{z}^2 \, dz.$$

SOLUTION: We write $C = C_1 + C_2 + C_3 + C_4$ as shown in the figure below, then

$$\int_C \overline{z}^2 \, dz = \int_{C_1} \overline{z}^2 \, dz + \int_{C_2} \overline{z}^2 \, dz + \int_{C_3} \overline{z}^2 \, dz + \int_{C_4} \overline{z}^2 \, dz.$$



On C_1: We have $z = t,\ 0 \le t \le 1$, and $\overline{z}^2 = t^2$, so that

$$\int_{C_1} \overline{z}^2 \, dz = \int_0^1 t^2 \, dt = \left.\frac{t^3}{3}\right|_0^1 = \frac{1}{3}.$$

On C_2: We have $z = 1 + it,\ 0 \le t \le 1$, and $\overline{z}^2 = (1 - it)^2 = (1 - t^2) - 2it$, so that

$$\int_{C_2} \overline{z}^2 \, dz = i\int_0^1 (1 - t^2) \, dt + 2\int_0^1 t \, dt = i\left(t - \frac{t^3}{3}\right)\Big|_0^1 + \left.\frac{2t^2}{2}\right|_0^1 = \frac{2i}{3} + 1.$$

On C_3: We have $z = t + i,\ 0 \le t \le 1$, and $\overline{z}^2 = (t - i)^2 = (t^2 - 1) - 2it$, so that

$$\int_{C_3} \overline{z}^2 \, dz = \int_1^0 (t^2 - 1) \, dt - 2i\int_1^0 t \, dt = \int_0^1 (1 - t^2) \, dt + 2i\int_0^1 t \, dt = \left(t - \frac{t^3}{3}\right)\Big|_0^1 + 2i\left.\frac{t^2}{2}\right|_0^1 = \frac{2}{3} + i.$$

On C_4: We have $z = it,\ 0 \le t \le 1$, and $\overline{z}^2 = (-it)^2 = -t^2$, so that

$$\int_{C_4} \overline{z}^2 \, dz = \int_1^0 (-t^2) \, i dt = i\int_0^1 t^2 \, dt = i\left.\frac{t^3}{3}\right|_0^1 = \frac{i}{3}.$$

Therefore,

$$\oint_C \overline{z}^2 \, dz = \frac{1}{3} + \frac{2i}{3} + 1 + \frac{2}{3} + i + \frac{i}{3} = 2 + 2i.$$

Question 11.

Evaluate

$$\oint_{|z|=1} \frac{\mathrm{Log}(z+2)}{z^2} \, dz.$$

(the circle $|z| = 1$ is oriented counterclockwise)

SOLUTION: The function $\text{Log}(z+2)$ is analytic at each point in $\mathbb{C}$ except on the portion of the real axis where $x \leq -2$. The integrand

$$f(z) = \frac{\text{Log}(z+2)}{z^2},$$

is analytic on and interior to the circle $|z| = 1$, except at the point $z = 0$, where it has a pole of order two with residue

$$B = \lim_{z\to 0} \frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0} \frac{d}{dz}\left[\text{Log}(z+2)\right] = \lim_{z\to 0} \frac{1}{z+2} = \frac{1}{2}.$$

From the Cauchy residue theorem, we have

$$\oint_{|z|=1} \frac{\text{Log}(z+2)}{z^2}\, dz = 2\pi i \cdot \frac{1}{2} = \pi i.$$

Question 12.

Evaluate

$$\oint_{|z|=2} \frac{\tan(z/2)}{(z-1)^2}\, dz.$$

(the circle $|z| = 2$ is oriented counterclockwise)

SOLUTION: The function

$$\tan(z/2) = \frac{\sin(z/2)}{\cos(z/2)}$$

is analytic everwhere except at the isolated zeros of $\cos(z/2)$, that is, at the points

$$\frac{z}{2} = \left(n + \frac{1}{2}\right)\pi, \quad n = \pm 1,\, \pm 2,\, \pm 3, \dot{s}$$

or

$$z = (2n+1)\pi, \quad n = \pm 1,\, \pm 2,\, \pm 3, \ldots,$$

all of which lie outside the circle $|z| = 2$.

Therefore, the function

$$\frac{\tan(z/2)}{(z-1)^2}$$

is analytic everywhere on and inside the circle $|z| = 2$, except at the point $z = 1$, where it has a double pole with residue

$$B = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right] = \lim_{z\to 1} \frac{d}{dz}\left[\tan(z/2)\right] = \lim_{z\to 1} \frac{1}{2}\sec^2(z/2) = \frac{1}{2}\sec^2(1/2).$$

From the Cauchy residue theorem, we have

$$\oint_{|z|=2} \frac{\tan(z/2)}{(z-1)^2}\, dz = \pi i \sec^2(1/2).$$

Question 13.

Evaluate

$$\int_i^{3+i} (z-1)^3\, dz.$$

SOLUTION: The function

$$f(z) = (z-1)^3$$

is entire, and has an antiderivative

$$F(z) = \frac{(z-1)^4}{4},$$

and

$$\int_i^{3+i} (z-1)^3\,dz = F(3+i) - F(i) = \frac{1}{4}\left[(2+i)^4 - (1-i)^4\right].$$

Question 14.

(a) Given functions $u(x,y)$ and $v(x,y)$ state sufficient conditions (on the partial derivatives) for

$$f(z) = u(x,y) + iv(x,y)$$

to be analytic at a point z_0.

(b) State the *Cauchy Integral Formula.*

SOLUTION:

(a) The function $f(z) = u(x,y) + iv(x,y)$ is **differentiable** at the point $z_0 = x_0 + iy_0$ if the first-order partial derivatives of u and v exist at each point of a neighborhood of the point (x_0, y_0), are continuous at the point (x_0, y_0), and satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

at the point (x_0, y_0).

However, the question asks for a sufficient condition for the function $f(z)$ to be **analytic** at the point z_0, and for this the derivative of $f(z)$ must exist at each point of a neighborhood of z_0. Therefore, a sufficient condition for $f(z)$ to be analytic at a point z_0 is that the first-order partial derivatives exist and are continuous throughout a neighborhood of (x_0, y_0), and satisfy the Cauchy-Riemann equations at each point in that neighborhood.

(b) The Cauchy Integral Formula states that

> If the function $f(z)$ is analytic everywhere inside and on a positively oriented simple closed contour $\mathcal{C}$, and if z_0 is any point interior to $\mathcal{C}$, then
>
> $$f(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{z - z_0}\,dz.$$

Question 15.

Obtain the first four (4) non-zero terms of the *Laurent series expansion* of the function

$$f(z) = \frac{1}{e^z - 1},$$

valid in the domain $0 < |z| < 2\pi$.

SOLUTION: The function

$$f(z) = \frac{1}{e^z - 1}$$

has a simple pole at the point $z_0 = 0$, since

$$g(z) = e^z - 1 = z\left(1 + \frac{1}{2!}z + \frac{1}{3!}z^2 + \frac{1}{4!}z^3 + \cdots\right)$$

has a zero of order 1 at $z_0 = 0$.

The residue of $f(z)$ at $z_0 = 0$ is

$$\operatorname*{Res}_{z=0}(f(z)) = \lim_{z\to 0}\frac{z}{e^z - 1} = \frac{1}{\displaystyle\lim_{z\to 0}\left(\frac{e^z - e^0}{z - 0}\right)} = 1$$

since the limit is just the reciprocal of the derivative of e^z evaluated at $z = 0$.

Suppose that the Laurent series expansion of $f(z)$ is

$$f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

valid for $0 < |z| < 2\pi$, then since $f(z)\cdot(e^z - 1) = 1$, we have

$$\left(\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n\right)\cdot\left(\sum_{n=1}^{\infty} z^n\right) = 1,$$

that is,

$$\left(1 + \sum_{n=0}^{\infty} a_n z^{n+1}\right)\cdot\left(\sum_{n=1}^{\infty} z^n\right) = z$$

for $0 < |z| < 2\pi$.

Therefore,

$$(1 + a_0 z + a_1 z^2 + a_2 z^3 + a_4 z^4 + \cdots)\left(z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots\right) = z,$$

and collecting terms that multiply z^k, for $k = 1, 2, 3, 4$, we have

$$z + \left(a_0 + \frac{1}{2!}\right)z^2 + \left(a_1 + \frac{a_0}{2!} + \frac{1}{3!}\right)z^3 + \left(a_2 + \frac{a_1}{2!} + \frac{a_0}{3!} + \frac{1}{4!}\right)z^4 + \left(a_3 + \frac{a_2}{2!} + \frac{a_1}{3!} + \frac{a_0}{4!} + \frac{1}{5!}\right)z^5 \cdots = z,$$

so that

$$a_0 + \frac{1}{2!} = 0$$

$$a_1 + \frac{a_0}{2!} + \frac{1}{3!} = 0$$

$$a_2 + \frac{a_1}{2!} + \frac{a_0}{3!} + \frac{1}{4!} = 0$$

$$a_3 + \frac{a_2}{2!} + \frac{a_1}{3!} + \frac{a_0}{4!} + \frac{1}{5!} = 0,$$

from which we can easily solve for a_0, a_1, a_2, a_3.

Question 16.

Let $f(z) = \dfrac{1}{q(z)^2}$ where $q(z)$ is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$.

Show that z_0 is a pole of order $m = 2$ of the function $f(z)$ with residue

$$b_1 = -\frac{q''(z_0)}{q'(z_0)^3}.$$

SOLUTION: Note first that z_0 is a zero of order $m = 1$ of the function $q(z)$, so that

$$q(z) = (z - z_0)g(z)$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

Then we can write

$$f(z) = \frac{\varphi(z)}{(z-z_0)^2},$$

where $\varphi(z) = 1/g(z)^2$ is analytic at z_0 and $\varphi(z_0) \neq 0$. Therefore, z_0 is a pole of order $m = 2$ of the function $f(z)$.

Now,

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = \varphi'(z) = -\frac{2g'(z)}{g(z)^3},$$

and

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = -\frac{2g'(z_0)}{g(z_0)^3} = -\frac{q''(z_0)}{q'(z_0)^3}, \qquad (*)$$

since

$$q'(z) = g(z) + (z-z_0)g'(z)$$

so that

$$q''(z) = 2g'(z) + (z-z_0)g''(z).$$

Therefore, $q''(z_0) = 2g'(z_0)$ and $q'(z_0) = g(z_0)$, from which $(*)$ follows.

Question 17.

Obtain the expansion of the function

$$f(z) = \frac{z^2+z+1}{z^3}$$

into its *Laurent series*, valid in the domain $0 < |z| < \infty$.

Solution: We have

$$f(z) = \left(z^2+z+1\right)\cdot\frac{1}{z^3} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$$

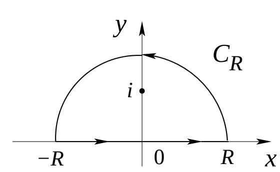
for $|z| > 0$, and this is the Laurent series expansion for $f(z)$ in the domain $0 < |z| < \infty$.

Question 18.

Using residues, show that

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{\pi}{2}.$$

Solution: Let $f(z) = \dfrac{z^2}{(1+z^2)^2}$, and for $R > 1$, consider the integral of f over the contour C_R shown below.



We have

$$\int_{C_R} \frac{z^2dz}{(1+z^2)^2} + \int_{-R}^{R} \frac{x^2dx}{(1+x^2)^2} = 2\pi i \operatorname*{Res}_{z=i}\left(\frac{z^2}{(1+z^2)^2}\right).$$

Now,

$$f(z) = \frac{z^2}{(1+z^2)^2} = \frac{\Phi(z)}{(z-i)^2}$$

where $\Phi(z) = \dfrac{z^2}{(z+i)^2}$ is analytic at $z = i$ and $\Phi(i) = \dfrac{1}{4} \neq 0$, so that f has a pole of order $m = 2$ at $z = i$, with

$$\operatorname*{Res}_{z=i}(f(z)) = \Phi'(i) = \frac{2z}{(z+i)^2}\bigg|_{z=i} - \frac{2z^2}{(z+i)^3}\bigg|_{z=i} = \frac{2i}{(2i)^2} + \frac{2}{(2i)^3} = -\frac{2}{(2i)^3} = -\frac{i}{4}.$$

Therefore,

$$\int_{C_R} \frac{z^2 dz}{(1+z^2)^2} + \int_{-R}^{R} \frac{x^2 dx}{(1+x^2)^2} = 2\pi i \left(-\frac{i}{4}\right) = \frac{\pi}{2} \qquad (*)$$

However, on C_R, we have $z = Re^{i\theta}$, and

$$|1+z^2|^2 \geq \left(|z|^2 - 1\right)^2 \geq (R^2-1)^2$$

if $R > 1$, so that

$$\left|\int_{C_R} \frac{z^2 dz}{(1+z^2)^2}\right| \leq \frac{R^2 \cdot 2\pi R}{(R^2-1)^2} \to 0$$

as $R \to \infty$.

Letting $R \to \infty$ in $(*)$, we have

$$\int_{-\infty}^{\infty} \frac{x^2\, dx}{(1+x^2)^2} = \frac{\pi}{2}.$$

Question 19.

(a) State the Cauchy integral formula for the n^{th} derivative $f^{(n)}(z_0)$ of a function $f(z)$ which is analytic everywhere inside and on a simple closed contour C (described in the positive sense) and z_0 is any point interior to C.

(b) Use the Cauchy integral formula to evaluate the integral

$$\int_{|z|=1} \frac{\sin z}{z^8}\, dz$$

where the circle $|z| = 1$ is traversed in the counterclockwise direction.

SOLUTION:

(a) The Cauchy integral formula for derivatives states that if $f(z)$ is analytic everywhere inside and on a simple closed positively oriented contour C, and z_0 is any point interior to c, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

(b) If we take $f(z) = \sin z$, and $C : |z| = 1$, then the Cauchy integral formula with $z_0 = 0$ and $n = 7$ says that

$$\frac{7!}{2\pi i} \int_{|z|=1} \frac{\sin z}{z^8}\, dz = \frac{d^7 \sin z}{dz^7}\bigg|_{z=0} = -\cos 0 = -1,$$

and

$$\int_{|z|=1} \frac{\sin z}{z^8}\, dz = -\frac{2\pi i}{7!}$$

where the circle $|z| = 1$ is traversed in the counterclockwise direction.

Question 20.

Find the Laurent series expansion of the function

$$f(z) = \frac{z}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)$$

valid on the following annular domains.

(a) $0 < |z-1| < 2$.

(b) $0 < |z+1| < 2$.

Solution:

(a) We expand the function in a Laurent series expansion about the point $z_0 = 1$:

$$\begin{aligned}
\frac{z}{z^2-1} &= \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)\\
&= \frac{1}{2}\frac{1}{z-1} + \frac{1}{2}\frac{1}{z-1+2}\\
&= \frac{1}{2}\frac{1}{z-1} + \frac{1}{4}\frac{1}{1+\dfrac{z-1}{2}}\\
&= \frac{1}{2}\frac{1}{z-1} + \frac{1}{4}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{2^2} - \frac{(z-1)^3}{2^3} + \cdots\right]
\end{aligned}$$

so that

$$\frac{z}{z^2-1} = \frac{1}{2}\frac{1}{z-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(z-1)^n$$

valid for $0 < |z-1| < 2$.

(b) We expand the function in a Laurent series expansion about the point $z_0 = -1$:

$$\begin{aligned}
\frac{z}{z^2-1} &= \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right)\\
&= \frac{1}{2}\frac{1}{z+1} + \frac{1}{2}\frac{1}{z+1-2}\\
&= \frac{1}{2}\frac{1}{z+1} - \frac{1}{4}\frac{1}{1-\dfrac{z+1}{2}}\\
&= \frac{1}{2}\frac{1}{z+1} - \frac{1}{4}\left[1 + \frac{(z+1)}{2} + \frac{(z+1)^2}{2^2} + \frac{(z+1)^3}{2^3} + \cdots\right]
\end{aligned}$$

so that

$$\frac{z}{z^2-1} = \frac{1}{2}\frac{1}{z+1} - \sum_{n=0}^{\infty}\frac{1}{2^{n+2}}(z+1)^n$$

valid for $0 < |z+1| < 2$.

Question 21.

Find and classify (according to the terms *pole, removable, essential*) the singular points of

$$f(z) = \frac{z}{1-\cos z}.$$

For each pole, give its order and compute the residue there.

Solution: Note that $f(z) = \dfrac{z}{1-\cos z}$ has isolated singular points at $z = 2\pi n,\ n \in \mathbb{Z}$, and that for $n = 0, \pm 1, \pm 2, \ldots,$ we have

$$\cos(z - 2\pi n) = \cos z,$$

so that if $g(z) = 1 - \cos z$, then

$$g(z) = 1 - \cos(z - 2\pi n) = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k (z-2\pi n)^{2k}}{(2k)!},$$

that is,

$$g(z) = (z-2\pi n)^2 \cdot \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(z-2\pi n)^{2(k-1)}}{(2k)!} = (z-2\pi n)^2 \cdot \phi(z),$$

where

$$\phi(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(z-2\pi n)^{2(k-1)}}{(2k)!}$$

is analytic at $z = 2\pi n$ and $\phi(2\pi n) = 1 \neq 0$.

Therefore, $g(z) = 1 - \cos z$ has a zero of order 2 at $z = 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots,$ and so

$$f(z) = \frac{z}{1-\cos z}$$

has a **simple pole** at $z = 0$, and a **pole of order 2** at $z = 2\pi n$ for $n = \pm 1, \pm 2, \ldots.$

For the simple pole at $z = 0$,

$$f(z) = \frac{z}{1-\cos z} = \frac{1}{z} \cdot \frac{1}{\frac{1}{2!} - \frac{1}{4!}z^2 + \cdots} = \frac{\Phi_0(z)}{z},$$

and $\operatorname*{Res}_{z=0}\left(\dfrac{z}{1-\cos z}\right) = 2.$

For the pole of order 2 at $z = 2\pi n$,

$$f(z) = \frac{z}{1-\cos(z-2\pi n)} = \frac{1}{(z-2\pi n)^2} \cdot \frac{z}{\frac{1}{2!} - \frac{1}{4!}(z-2\pi n)^2 + \cdots} = \frac{\Phi_1(z)}{(z-2\pi n)^2},$$

and

$$\operatorname*{Res}_{z=2\pi n}\left(\frac{z}{1-\cos z}\right) = \lim_{z\to 2n\pi} \frac{1}{1!} \cdot \frac{d}{dz}\left[(z-2n\pi)^2 f(z)\right] = \frac{\Phi_1'(2\pi n)}{1!} = 2$$

for $n = \pm 1, \pm 2, \ldots.$

Question 22.

Find the Laurent expansion of $f(z) = \dfrac{z}{(z-1)(z-2)}$ valid in the domain

(a) $0 < |z-1| < 1,$

(b) $0 < |z-2| < 1,$

(c) $1 < |z| < 2.$

SOLUTION: Note that

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$$

for $z \neq 1, 2$.

(a) For $0 < |z-1| < 1$, we have

$$\frac{1}{z-2} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n,$$

so that

$$f(z) = -\frac{1}{z-1} - 2\sum_{n=0}^{\infty}(z-1)^n,$$

valid for $0 < |z-1| < 1$.

(b) For $0 < |z-2| < 1$, we have

$$\frac{1}{z-1} = \frac{1}{1+(z-2)} = \sum_{n=0}^{\infty}(-1)^n(z-2)^n,$$

so that

$$f(z) = \frac{2}{z-2} - \sum_{n=0}^{\infty}(-1)^n(z-2)^n,$$

valid for $0 < |z-2| < 1$.

(c) For $1 < |z| < 2$, we have

$$f(z) = \frac{2}{z-2} - \frac{1}{z-1} = -\frac{1}{1-z/2} - \frac{1}{z}\cdot\frac{1}{1-1/z},$$

and

$$f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \sum_{n=1}^{\infty}\frac{1}{z^n},$$

valid for $1 < |z| < 2$.

Question 23.

For $n = 1, 2, \ldots$ find the $2n^{\text{th}}$ derivatives of

$$f(z) = \sin\left(z^2\right)$$

at $z = 0$ by using the Cauchy integral formula for derivatives.

SOLUTION: We have

$$f(z) = \sin(z^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (z^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{(2k+1)!},$$

valid for all $z \in \mathbb{C}$, and from the Cauchy integral formula,

$$f^{(2n)}(0) = \frac{(2n)!}{2\pi i} \oint_{|z|=1} \frac{\sin(z^2)}{z^{2n+1}}\, dz = (2n)! \cdot \operatorname*{Res}_{z=0} \left(\frac{\sin(z^2)}{z^{2n+1}} \right).$$

If n is even, then $2n = 4m$ for some integer $m \geq 0$, and

$$\frac{\sin(z^2)}{z^{4m+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{4(k-m)+1},$$

so that $\operatorname*{Res}_{z=0} \left(\frac{\sin(z^2)}{z^{4m+1}} \right) = 0$, and $f^{(4m)}(0) = 0$.

If n is odd, then $2n = 4m + 2$ for some integer $m \geq 0$, and

$$\frac{\sin(z^2)}{z^{4m+3}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{4(k-m)-1},$$

so that $\operatorname*{Res}_{z=0} \left(\frac{\sin(z^2)}{z^{4m+3}} \right) = \frac{(-1)^m}{(2m+1)!}$, and $f^{(4m+2)}(0) = \frac{(-1)^m \cdot (4m+2)!}{(2m+1)!}$.

Question 24.

(a) Show that

$$\cos\theta > 1 - \frac{2}{\pi}\theta$$

for $0 < \theta < \frac{\pi}{2}$.

Hint: Look at the graphs of $y = \cos\theta$ and $y = 1 - \frac{2}{\pi}\theta$ on the interval $0 < \theta < \frac{\pi}{2}$.

(b) From part (a), show that

$$\int_0^{\frac{\pi}{2}} e^{-R\cos\theta}\, d\theta < \frac{\pi}{2R}\left(1 - e^{-R}\right)$$

for $R > 0$.

SOLUTION:

(a) Define the function

$$f(\theta) = \cos\theta - \left(1 - \frac{2}{\pi}\theta\right)$$

for $0 < \theta < \frac{\pi}{2}$, then

$$f'(\theta) = -\sin\theta + \frac{2}{\pi} \qquad \text{and} \qquad f''(\theta) = -\cos\theta$$

so that $f''(\theta) < 0$ for $0 < \theta < \frac{\pi}{2}$.

Since

$$f(0) = 0 \qquad \text{and} \qquad f\left(\frac{\pi}{2}\right) = 0,$$

and f is strictly concave downward on the interval, then

$$f(x) = \cos\theta - \left(1 - \frac{2}{\pi}\,\theta\right) > 0$$

for all $0 < \theta < \dfrac{\pi}{2}$.

(b) For $R > 0$, we have

$$e^{-R\cos\theta} < e^{-R\left(1-\frac{2}{\pi}\theta\right)},$$

and therefore

$$\int_0^{\frac{\pi}{2}} e^{-R\cos\theta}\,d\theta < \int_0^{\frac{\pi}{2}} e^{-R\left(1-\frac{2}{\pi}\theta\right)}\,d\theta = e^{-R}\int_0^{\frac{\pi}{2}} e^{\frac{2R}{\pi}\theta}\,d\theta = e^{-R}\frac{\pi}{2R}\,e^{\frac{2R}{\pi}\theta}\Big|_0^{\frac{\pi}{2}} = e^{-R}\frac{\pi}{2R}\left(e^R - 1\right),$$

that is,

$$\int_0^{\frac{\pi}{2}} e^{-R\cos\theta}\,d\theta < \frac{\pi}{2R}\left(1 - e^{-R}\right).$$

Question 25.

(a) Show that the function

$$f(z) = \begin{cases} \dfrac{\operatorname{Log}(z+1)}{z}, & \text{for} \quad 0 < |z| < 1 \\ 1, & \text{for} \quad z = 0 \end{cases}$$

is analytic at $z = 0$.

(b) Find a formula for the derivatives $f^{(n)}(0)$.

Solution:

(a) Let $g(z) = \operatorname{Log}(z+1)$, then $g(z)$ is analytic at all z for which

$$z + 1 = \rho e^{i\phi}, \quad \rho > 0,\ -\pi < \phi < \pi.$$

In particular, $g(z)$ is analytic in the disk $|z| < 1$, with

$$g'(z) = \frac{1}{z+1} = \sum_{n=0}^{\infty}(-1)^n z^n$$

for $|z| < 1$.

If $|z| < 1$ and C is any contour joining 0 to z which lies entirely inside the disk, then

$$\operatorname{Log}(z+1) - \operatorname{Log}(1) = \sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{n+1},$$

that is,

$$\operatorname{Log}(z+1) = \sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{n+1}$$

for $|z| < 1$, and therefore

$$\frac{\operatorname{Log}(z+1)}{z} = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n+1},$$

for $0 < |z| < 1$.

However, the series on the right-hand side converges to 1 for $z = 0$, so that if we define

$$f(z) = \begin{cases} \dfrac{\mathrm{Log}(z+1)}{z}, & \text{for} \quad 0 < |z| < 1 \\ 1, & \text{for} \quad z = 0 \end{cases}$$

then

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n+1}$$

for $|z| < 1$, and therefore f is analytic at $z = 0$.

(b) Since

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n+1}$$

for $|z| < 1$, then this is the Maclaurin series expansion for $f(z)$, and therefore

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{n+1},$$

that is,

$$f^{(n)}(0) = \frac{(-1)^n \cdot n!}{n+1}$$

for $n \geq 0$.

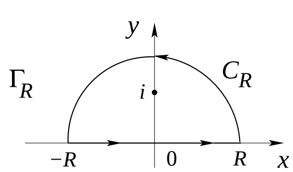
Question 26.

Using Cauchy's residue theorem, show that

$$\int_0^{\infty} \frac{\cos ax}{x^2+1}\, dx = \frac{\pi}{2} e^{-a}$$

for $a > 0$.

Solution: Consider the integral $\displaystyle\oint_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\, dz$ where Γ_R is the contour shown below, traversed in the counterclockwise direction.



For $R > 1$, the integrand $f(z) = \dfrac{e^{iaz}}{z^2+1}$ has a simple pole at $z_0 = i$ inside the contour Γ_R with residue

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{e^{-a}}{2i},$$

and from the Cauchy residue theorem

$$\oint_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\, dz = \int_{-R}^{0} \frac{e^{iax}}{x^2+1}\, dx + \int_0^R \frac{e^{iax}}{x^2+1}\, dx + \int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz = \pi\, e^{-a},$$

that is,

$$\int_0^R \frac{e^{-iax}}{x^2+1}\, dx + \int_0^R \frac{e^{iax}}{x^2+1}\, dx + \int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz = \pi\, e^{-a},$$

so that

$$2\int_0^R \frac{\cos ax}{x^2+1}\,dx + \int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz = \pi\, e^{-a}. \tag{$**$}$$

Now, on C_R we have

$$\left|e^{iaz}\right| = e^{-ay} \le 1$$

since $a > 0$ and $y \ge 0$, and from the triangle inequality

$$|z^2+1| \ge |z|^2 - 1 = R^2 - 1$$

for all $z \in C_R$, so that

$$\left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz\right| \le \frac{\pi R}{R^2-1} \longrightarrow 0$$

as $R \to \infty$.

Letting $R \to \infty$ in $(**)$ we get

$$2\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \pi\, e^{-a},$$

or

$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}\, e^{-a}.$$

Question 27.

Evaluate $\displaystyle\oint_{|z|=7} \frac{1+z}{1-\cos z}\,dz$.

Solution: The function $f(z) = \dfrac{1+z}{1-\cos z}$ has a pole of order 2 at $z = 2\pi n,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$, and since

$$\cos(z - 2\pi n) = \cos z,$$

we can write

$$f(z) = \frac{1+z}{1-\cos z} = \frac{\phi(z)}{(z-2\pi n)^2},$$

where

$$\phi(z) = \frac{z+1}{\frac{1}{2!} - \frac{1}{4!}(z-2\pi n)^2 + \cdots}$$

is analytic at $z = 2\pi n$, and $\phi(2\pi n) = \dfrac{2\pi n + 1}{\frac{1}{2!}} \ne 0$.

Now,

$$\operatorname*{Res}_{z=2\pi n}\left(\frac{z+1}{1-\cos z}\right) = \frac{\phi'(2\pi n)}{1!} = \left.\frac{1\cdot\left\{\frac{1}{2!} - \frac{1}{4!}(z-2\pi n)^2 + \cdots\right\} - (z+1)\cdot\left\{\frac{-2(z-2\pi n)}{4!} + \cdots\right\}}{\left\{\frac{1}{2!} - \frac{1}{4!}(z-2\pi n)^2 + \cdots\right\}^2}\right|_{z=2\pi n},$$

so that

$$\operatorname*{Res}_{z=2\pi n}\left(\frac{z+1}{1-\cos z}\right) = \frac{\frac{1}{2!}}{\left(\frac{1}{2!}\right)^2} = 2$$

for $n = 0,\ \pm 1,\ \pm 2,\ \ldots$.

The only singular points inside the contour $|z| = 7$ are $z = 2\pi n$ for $n = 0, \pm 1$, and therefore

$$\oint_{|z|=7} \frac{z+1}{1-\cos z}\,dz = 2\pi i\{2+2+2\} = 12\pi i.$$

Question 28.

Show that $\displaystyle\oint_{|z|=1} \frac{\operatorname{Log}(z+2)}{z}\,dz = 2\pi i \ln 2.$

Solution: Let $\Phi(z) = \operatorname{Log}(z+2)$, then $\Phi(z)$ is analytic for all z with

$$z + 2 = \rho e^{i\phi}, \quad \rho > 0,\ -\pi < \phi < \pi.$$

In particular, $\Phi(z)$ is analytic for $|z| < 1$, with $\Phi(0) = \operatorname{Log}(2) = \ln 2 \neq 0$, and therefore

$$f(z) = \frac{\operatorname{Log}(z+2)}{z} = \frac{\Phi(z)}{z}$$

has a simple pole at $z = 0$ with

$$\operatorname*{Res}_{z=0}\left(\frac{\operatorname{Log}(z+2)}{z}\right) = \Phi(0) = \ln 2,$$

and therefore

$$\oint_{|z|=1} \frac{\operatorname{Log}(z+2)}{z}\,dz = 2\pi i \ln 2.$$

Question 29.

Show that $\displaystyle\oint_{|z|=1} e^{\left(z+\frac{1}{z}\right)}\,dz = 2\pi i \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}.$

Solution: We have

$$f(z) = e^{\left(z+\frac{1}{z}\right)} = e^z \cdot e^{\frac{1}{z}} = \left(1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots\right)\left(1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right)$$

for $0 < |z| < \infty$.

To find $\operatorname*{Res}_{z=0}\big(e^{z+\frac{1}{z}}\big)$ we collect terms that multiply $\dfrac{1}{z}$, and we get

$$\operatorname*{Res}_{z=0}\big(e^{z+\frac{1}{z}}\big) = \frac{1}{1!} + \frac{1}{1!\cdot 2!} + \frac{1}{2!\cdot 3!} + \frac{1}{3!\cdot 4!} + \cdots,$$

that is,

$$\operatorname*{Res}_{z=0}\big(e^{z+\frac{1}{z}}\big) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!},$$

and therefore

$$\oint_{|z|=1} e^{\left(z+\frac{1}{z}\right)}\,dz = 2\pi i \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}.$$

Question 30.

Let 1, ω, and ω^2 be the three cube roots of 1, and let

$$f(z) = \cos(z) \cdot \cos(\omega z) \cdot \cos(\omega^2 z).$$

Clearly, $f(z)$ is an entire function.

(a) Show that

$$f(z) = f(\omega z) = f(\omega^2 z).$$

(b) Show that

$$3f(z) = f(z) + f(\omega z) + f(\omega^2 z).$$

(c) Show that the Maclaurin series of f has nonzero coefficients a_n only when n is a multiple of 3.

SOLUTION:

(a) We have

$$f(\omega z) = \cos(\omega z) \cdot \cos(\omega^2 z) \cdot \cos(\omega^3 z),$$

and since $\omega^3 = 1$, then

$$f(\omega z) = \cos(\omega z) \cdot \cos(\omega^2 z) \cdot \cos(z) = f(z).$$

Similarly,

$$f(\omega^2 z) = \cos(\omega^2 z) \cdot \cos(\omega^3 z) \cdot \cos(\omega^4 z),$$

and since $\omega^3 = 1$, and $\omega^4 = \omega$, then

$$f(\omega^2 z) = \cos(\omega^2 z) \cdot \cos(z) \cdot \cos(\omega z) = f(z).$$

(b) Therefore,

$$3f(z) = f(z) + f(\omega z) + f(\omega^2 z)$$

for all $z \in \mathbb{C}$.

(c) Since the function $f(z)$ is entire, it has a Maclaurin series expansion that converges for all $z \in \mathbb{C}$, and we can write

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots = \sum_{n=0}^{\infty} a_n z^n,$$

and from part (b),

$$3f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} a_n \omega^n z^n + \sum_{n=0}^{\infty} a_n \omega^{2n} z^n,$$

that is,

$$3f(z) = \sum_{n=0}^{\infty} \left(1 + \omega^n + \omega^{2n}\right) a_n z^n.$$

Now, if $n = 3k$, where k is a nonnegative integer, then

$$1 + \omega^n + \omega^{2n} = 1 + (\omega^3)^k + (\omega^3)^{2k} = 1 + 1 + 1 = 3 \neq 0.$$

If $n = 3k + 1$, where k is a nonnegative integer, then

$$1 + \omega^n + \omega^{2n} = 1 + (\omega^3)^k \cdot \omega + (\omega^3)^{2k} \cdot \omega^2 = 1 + \omega + \omega^2 = 0.$$

If $n = 3k + 2$, where k is a nonnegative integer, then

$$1 + \omega^n + \omega^{2n} = 1 + (\omega^3)^k \cdot \omega^2 + (\omega^3)^{2k} \cdot \omega^4 = 1 + (\omega^3)^k \cdot \omega^2 + (\omega^3)^{2k} \cdot \omega = 1 + \omega^2 + \omega = 0.$$

Therefore, the Maclaurin series for f has nonzero coefficents only if n is a multiple of 3.

Question 31.

If p and q are real numbers with $p > q > 0$, show that

$$\int_0^{2\pi} \frac{d\theta}{(p + q\cos\theta)^2} = \frac{2\pi p}{(p^2 - q^2)^{\frac{3}{2}}}.$$

Solution: We make the substitution $z = e^{i\theta}$, and convert the integral into a contour integral around the unit circle $|z| = 1$.

$$I = \int_0^{2\pi} \frac{d\theta}{(p + q\cos\theta)^2} = \oint_{|z|=1} \frac{dz}{iz\{p + \frac{q}{2}\left(z + \frac{1}{z}\right)\}^2} = -i\oint_{|z|=1} \frac{z\,dz}{\{\frac{q}{2}z^2 + pz + \frac{q}{2}\}^2}.$$

The integrand in the integral on the right has poles of order 2 at each of the points

$$z_1 = \frac{1}{q}\left[-p + \sqrt{p^2 - q^2}\right] \quad \text{and} \quad z_2 = \frac{1}{q}\left[-p - \sqrt{p^2 - q^2}\right].$$

Since we assumed that $p > q > 0$, the pole z_2 lies outside the unit circle. And since

$$\frac{q}{2}z^2 + pz + \frac{q}{2} = \frac{q}{2}(z - z_1)(z - z_2),$$

the product of the roots is 1, so that the pole z_1 lies inside the unit circle.

The residue b_1 of the pole of order 2 at the pole z_1 is given by

$$\begin{aligned} b_1 &= \lim_{z\to z_1}\left[\frac{d}{dz}\left\{\frac{z(z - z_1)^2}{(q^2/4)(z - z_1)^2(z - z_2)^2}\right\}\right] \\ &= \frac{4}{q^2}\left[\frac{d}{dz}\frac{z}{(z - z_2)^2}\right]_{z=z_1} \\ &= -\frac{4}{q^2}\frac{z_1 + z_2}{(z_1 - z_2)^3} \\ &= \frac{p}{(p^2 - q^2)^{3/2}}. \end{aligned}$$

Therefore, by Cauchy's Residue Theorem, we have

$$\int_0^{2\pi} \frac{d\theta}{(p + q\cos\theta)^2} = -i2\pi i b_1 = \frac{2\pi p}{(p^2 - q^2)^{3/2}}$$

for $p > q > 0$.

Question 32. Evaluate the integral

$$I = \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2}\,dx$$

where a and b are nonnegative real numbers. Integrals of this type are called **Frullani integrals**.

Note: The integral I **cannot** be represented as the difference of the integrals

$$\int_0^{\infty} \frac{\cos ax}{x^2}\,dx \quad \text{and} \quad \int_0^{\infty} \frac{\cos bx}{x^2}\,dx,$$

since both these integrals diverge.

To see this, since $\cos ax \to 1$ as $x \to 0^+$, there is a positive number η such that $|\cos ax| \geq 1/2$ for $0 < x < \eta$, and therefore

$$\int_\epsilon^\eta \frac{\cos ax}{x^2}\,dx \geq \int_\epsilon^\eta \frac{1}{2x^2}\,dx = \frac{1}{2\epsilon} - \frac{1}{2\eta}$$

whenever $0 < \epsilon < \eta$. Therefore, if we break the range of integration at η,

$$\int_\epsilon^\infty \frac{\cos ax}{x^2}\,dx = \int_\epsilon^\eta \frac{\cos ax}{x^2}\,dx + \int_\eta^\infty \frac{\cos ax}{x^2}\,dx, \tag{$*$}$$

but

$$\lim_{\epsilon\to 0^+} \int_\epsilon^\eta \frac{\cos ax}{x^2}\,dx = +\infty$$

so that the improper integral

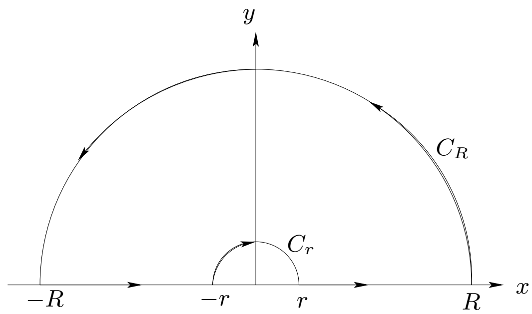
$$\int_0^\infty \frac{\cos ax}{x^2}\,dx$$

diverges (it converges if and only if **both** integrals on the right-hand side of $(*)$ converge).

Solution: We consider the function

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}$$

and take the integral of $f(z)$ over the indented contour shown below.



We get

$$\int_{-R}^{-r} f(x)\,dx + \int_{C_r} f(z)\,dz + \int_r^R f(z)\,dz + \int_{C_R} f(z)\,dz = 0. \tag{$**$}$$

Along C_R, we have

$$|e^{iaz}| = e^{-ay} \leq 1 \qquad \text{and} \qquad |e^{ibz}| = e^{-by} \leq 1,$$

since $a \geq 0$ and $b \geq 0$, and therefore

$$|f(z)| \leq \frac{2}{R^2}$$

for all z on C_R, so that

$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{2}{R^2} \cdot \pi R \longrightarrow 0$$

as $R \to \infty$.

To find a bound for the integral along C_r, we look at the Laurent expansion of $f(z)$ about 0 valid for $0 < |z| < \infty$. We have

$$f(z) = \frac{1}{z^2}\left\{ i(a-b)z + \frac{(iaz)^2 - (ibz)^2}{2!} + \cdots \right\} = \frac{i(a-b)}{z} + P(z),$$

where $P(z)$ is analytic at $z = 0$. Integrating over the semicircle C_r, we have

$$\int_{C_r} f(z)\,dz = i(a-b)\int_{C_r} \frac{dz}{z} + \int_{C_r} P(z)\,dz$$

and hence,

$$\lim_{r\to 0^+} \int_{C_r} f(z)\,dz = \lim_{r\to 0^+} \int_{C_r} \frac{dz}{z} + \lim_{r\to 0^+} \int_{C_r} P(z)\,dz.$$

Now, $P(z)$ is analytic at 0 and so is bounded in a neighborhood of $z = 0$, say $|P(z)| \leq M$ for all z with $|z| < \epsilon$, so by the $M\,L$ Theorem,

$$\left| \int_{C_r} P(z)\, dz \right| \leq M \cdot \pi r,$$

for $r < \epsilon$. Therefore the integral of $P(z)$ over C_r goes to 0 as $r \to 0^+$.

If we parametrize C_r as $z = re^{i\theta}$ for $0 \leq \theta \leq \pi$, then we have

$$\int_{C_r} \frac{1}{z}\, dz = \int_{\pi}^{0} \frac{ire^{i\theta}}{re^{i\theta}}\, d\theta == -\int_0^{\pi} i\, d\theta = -\pi i,$$

so that

$$\lim_{r \to 0^+} \int_{C_r} f(z)\, dz = i(a-b)(-\pi i) = (a-b)\pi.$$

Equating real and imaginary parts in $(**)$, and letting $r \to 0^+$ and $R \to \infty$, we have

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2}\, dx + (a-b)\pi = 0.$$

Since the integrand is an even function, this implies that

$$\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2}\, dx = \frac{b-a}{2}\pi.$$

Question 33.

Find the inverse Laplace transform of the function

$$F(s) = \frac{1}{(s^2+a^2)(s^2+b^2)},$$

where a and b are real numbers, and $a \neq b$. What happens if $a = b$?

Solution: The function $F(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}$ has simple poles at $s_1 = ai$, $s_2 = -ai$, $s_3 = bi$, and $s_4 = -bi$, and the inverse Laplace transform is

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{(s^2+a^2)(s^2+b^2)}\, ds = \sum_{k=1}^{4} \operatorname*{Res}_{s=s_k} \left[\frac{e^s t}{(s^2+a^2)(s^2+b^2)} \right],$$

where $\gamma > \max\{|s_1|, |s_2|, |s_3|, |s_4|\}$.

Now,

$$\operatorname*{Res}_{s=ai} \left[\frac{e^{st}}{(s^2+a^2)(s^2+b^2)} \right] = \lim_{s \to ai} \frac{e^{st}}{(s+ai)(s^2+b^2)} = \frac{e^{ait}}{2ai(b^2-a^2)},$$

and

$$\operatorname*{Res}_{s=-ai} \left[\frac{e^{st}}{(s^2+a^2)(s^2+b^2)} \right] = \lim_{s \to -ai} \frac{e^{st}}{(s-ai)(s^2+b^2)} = \frac{-e^{-ait}}{2ai(b^2-a^2)},$$

and

$$\operatorname*{Res}_{s=bi} \left[\frac{e^{st}}{(s^2+a^2)(s^2+b^2)} \right] = \lim_{s \to bi} \frac{e^{st}}{(s^2+a^2)(s+bi)} = \frac{e^{bit}}{2bi(a^2-b^2)},$$

and

$$\operatorname*{Res}_{s=-bi} \left[\frac{e^{st}}{(s^2+a^2)(s^2+b^2)} \right] = \lim_{s \to -bi} \frac{e^{st}}{(s^2+a^2)(s-bi)} = \frac{-e^{-bit}}{2bi(a^2-b^2)}.$$

Therefore, if $a \neq b$, then

$$f(t) = \frac{1}{b^2-a^2} \left[\frac{\sin at}{a} - \frac{\sin bt}{b} \right] \tag{$*$}$$

for $t > 0$.

If $a = b$, then $F(s)$ becomes

$$F(s) = \frac{1}{(s^2 + a^2)^2} = \frac{1}{(s - ai)^2 (s + ia)^2},$$

which has a pole of order $m = 2$ at $s = ai$ and a pole of order 2 at $s = -ai$.

For $s = ai$, the residue of $e^{st}F(s)$ is given by

$$\begin{aligned}
\operatorname*{Res}_{s=ai}(e^{st}F(s)) &= \lim_{s \to ai} \frac{d}{ds}\left[\frac{e^{st}}{(s + ai)^2}\right] \\
&= \lim_{s \to ai} \left[\frac{te^{st}}{(s + ai)^2} - \frac{2e^{st}}{(s + ia)^3}\right] \\
&= -\frac{te^{ait}}{4a^2} + \frac{2e^{ait}}{8a^3 i},
\end{aligned}$$

and

$$\operatorname*{Res}_{s=ai}(e^{st}F(s)) = -\frac{te^{ait}}{4a^2} + \frac{e^{ait}}{4a^3 i}.$$

For $s = -ai$, the residue of of $e^{st}F(s)$ is given by

$$\begin{aligned}
\operatorname*{Res}_{s=-ai}(e^{st}F(s)) &= \lim_{s \to -ai} \frac{d}{ds}\left[\frac{e^{st}}{(s - ai)^2}\right] \\
&= \lim_{s \to -ai} \left[\frac{te^{st}}{(s - ai)^2} - \frac{2e^{st}}{(s - ia)^3}\right] \\
&= -\frac{te^{-ait}}{4a^2} - \frac{2e^{-ait}}{8a^3 i},
\end{aligned}$$

and

$$\operatorname*{Res}_{s=-ai}(e^{st}F(s)) = -\frac{te^{-ait}}{4a^2} - \frac{e^{-ait}}{4a^3 i}.$$

Therefore, the inverse Laplace transform of $F(s)$ is given by

$$L^{-1}(F(s)) = \operatorname*{Res}_{s=ai}(e^{st}F(s)) + \operatorname*{Res}_{s=-ai}(e^{st}F(s)) = -\frac{te^{ait}}{4a^2} - \frac{te^{-ait}}{4a^2} + \frac{e^{ait}}{4a^3 i} - \frac{e^{-ait}}{4a^3 i},$$

That is,

$$L^{-1}(F(s)) = -\frac{1}{2a^2} t \cos at + \frac{1}{2a^3} \sin at,$$

for $t > 0$. You can check this by taking the Laplace transform of the right-hand side.