Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers

The Derivative and Cauchy-Riemann Equations in Polar Coordinates

Department of Mathematical and Statistical Sciences
University of Alberta

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In this note we will show that if

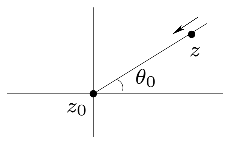
$$f(z) = u(r, \theta) + iv(r, \theta), \qquad \text{for } z = re^{i\theta}$$

is differentiable at a point $z_0 = r_0 e^{i\theta_0}$, then the derivative $f'(z_0)$ can be written in planar polar coordinates as

$$f'(z_0) = e^{-i\theta_0}\left(\frac{\partial u}{\partial r}(r_0, \theta_0) + i\frac{\partial v}{\partial r}(r_0, \theta_0)\right).$$

Suppose that $f'(z_0)$ exists at $z_0 = r_0 e^{i\theta_0}$, since the limit of the difference quotients doesn't depend on the direction from which $z \to z_0$, we consider two different approaches.

- If $z \to z_0$ along the ray $\arg(z) = \theta_0$, as in the figure,



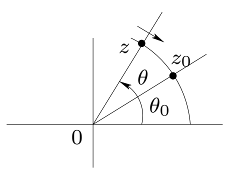
then $\Delta z = z - z_0 = (r - r_0)e^{i\theta_0}$, and

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ & \lim_{r \to r_0} \frac{u(r, \theta_0) + iv(r, \theta_0) - u(r_0, \theta_0) - v(r_0, \theta_0)}{(r - r_0)e^{i\theta_0}} \\ &= e^{-i\theta_0} \lim_{r \to r_0} \frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0} + ie^{-i\theta_0} \lim_{r \to r_0} \frac{v(r, \theta_0) - v(r_0, \theta_0)}{r - r_0}. \end{aligned}$$

Therefore,

$$f'(z_0) = e^{-\theta_0}\left(\frac{\partial u}{\partial r}(r_0, \theta_0) + i\frac{\partial v}{\partial r}(r_0, \theta_0)\right) \qquad (*)$$

- If $z \to z_0$ along the circle $|z| = r_0$, as in the figure,



then $\Delta z = z - z_0 = r_0 \left(e^{i\theta} - e^{i\theta_0}\right)$, and

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} + i \lim_{\theta \to \theta_0} \frac{v(r_0, \theta) - v(r_0, \theta_0)}{r_0(e^{i\theta} - e^{i\theta_0})} \\ &= \lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{r_0(\theta - \theta_0)} \left(\frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}\right) + i \lim_{\theta \to \theta_0} \frac{v(r_0, \theta) - v(r_0, \theta_0)}{r_0(\theta - \theta_0)} \left(\frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}\right). \end{aligned}$$

From l'Hospital's rule

$$\lim_{\theta - \theta_0} \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}} = \frac{1}{ie^{i\theta_0}},$$

so that

$$\begin{aligned} f'(z_0) &= \frac{1}{ir_0 e^{i\theta_0}} \frac{\partial u}{\partial \theta}(r_0, \theta_0) + \frac{i}{ir_0 e^{i\theta_0}} \frac{\partial v}{\partial \theta}(r_0, \theta_0) \\ &= -ie^{-i\theta_0} \frac{1}{r_0} \frac{\partial u}{\partial \theta}(r_0, \theta_0) + e^{-i\theta_0} \frac{1}{r_0} \frac{\partial v}{\partial \theta}(r_0, \theta_0). \end{aligned}$$

Therefore,

$$f'(z_0) = e^{-i\theta_0} \left(\frac{1}{r_0} \frac{\partial v}{\partial \theta}(r_0, \theta_0) - \frac{i}{r_0} \frac{\partial u}{\partial \theta}(r_0, \theta_0)\right). \qquad (**)$$

Equating real and imaginary parts in $(*)$ and $(**)$, we obtain the **Cauchy-Riemann equations** in polar form:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{1}{r} \frac{\partial u}{\partial \theta} &= -\frac{\partial v}{\partial r}. \end{aligned}$$