Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers

The Triangle Inequality in $\mathbb{C}$

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Theorem. Let $z_1, z_2 \in \mathbb{C}$, then

$$|z_1 + z_2| \le |z_1| + |z_2|,$$

and equality holds, that is, $|z_1 + z_2| = |z_1| + |z_2|$, if and only if one of z_1 or z_2 is a nonnegative multiple of the other.

Proof. First note that for any $\xi = \alpha + i\beta$ in $\mathbb{C}$, we have

$$|\xi| = \sqrt{\alpha^2 + \beta^2} \ge \alpha = \mathrm{Re}(\xi).$$

Now, if z_1 and z_2 are arbitrary complex numbers, and if $z_1 + z_2 = 0$, the result is obviously true, so we assume that $z_1 + z_2 \ne 0$, then

$$\begin{aligned}\frac{|z_1| + |z_2|}{|z_1 + z_2|} &= \frac{|z_1|}{|z_1 + z_2|} + \frac{|z_2|}{|z_1 + z_2|} = \left|\frac{z_1}{z_1 + z_2}\right| + \left|\frac{z_2}{z_1 + z_2}\right| \ge \mathrm{Re}\left(\frac{z_1}{z_1 + z_2}\right) + \mathrm{Re}\left(\frac{z_2}{z_1 + z_2}\right)\\ &= \mathrm{Re}\left(\frac{z_1}{z_1 + z_2} + \frac{z_2}{z_1 + z_2}\right) = \mathrm{Re}\ 1 = 1.\end{aligned}$$

Thus, $|z_1 + z_2| \le |z_1| + |z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

We give an alternate proof of the triangle inequality as follows:

$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})\\ &= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}\\ &= |z_1|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2,\end{aligned}$$

and since

$$\mathrm{Re}(z_1\overline{z_2}) \le |z_1\overline{z_2}| = |z_1|\,|z_2|,$$

then

$$|z_1 + z_2|^2 \le |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2 = (|z_1| + |z_2|)^2.$$

Taking nonnegative square roots, we have $|z_1 + z_2| \le |z_1| + |z_2|$.

In the above proof, equality holds if and only if $\mathrm{Re}(z_1\overline{z_2}) = |z_1\overline{z_2}|$, that is, $z_1\overline{z_2}$ is **real** and **nonnegative**, say $z_1\overline{z_2} = k$, then

$$z_1|z_2|^2 = kz_2,$$

and

$$z_1 = \frac{k}{|z_2|^2}\,z_2.$$

$\square$

Corollary. If a, b and z are points in the complex plane, then equality holds in the triangle inequality

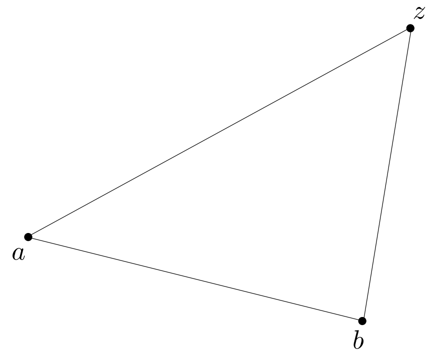
$$|a-b| \leq |a-z| + |z-b|$$

if and only if a, b and z are collinear and z lies between a and b.

Proof. Let a, b and z be arbitrary points in the complex plane, then from the previous theorem, in the triangle whose verices are the points a, b, and z, the triangle inequality can be written

$$|a-b| \leq |a-z| + |z-b|$$

by replacing z_1 by $a-z$ and z_2 by $z-b$.



Equality holds if and only if

$$z-b = \lambda(a-z) \qquad \text{or} \qquad a-z = \lambda(z-b)$$

for some $\lambda \geq 0$.

We assume without loss of generality that

$$z-b = \lambda(a-z)$$

for some $\lambda \geq 0$, so that

$$(1+\lambda)z = b + \lambda a,$$

and therefore

$$z = a + \frac{1}{1+\lambda}(b-a)$$

where $0 \leq \dfrac{1}{1+\lambda} \leq 1$.

The line passing through a and b is

$$L = \{\, \xi \in \mathbb{C} \,|\, \xi = a + \mu(b-a),\ \mu \in \mathbb{R} \,\},$$

and a, b and z are collinear (they all lie on L), and since $\mu = \dfrac{1}{1+\lambda} \in [0,1]$, then z lies between a and b.

$\square$

Exercise. Show that

$$\big|\,|z_1| - |z_2|\,\big| \leq |z_1 - z_2|$$

for all z_1, $z_2 \in \mathbb{C}$. When does equality hold?