MATH 309

MIDTERM EXAMINATION II SOLUTIONS

DATE: Friday July 14, 2017 TIME: 50 Minutes

Question 1. (10 points) Show that if $\Gamma = \{ z \in \mathbb{C} \,|\, |z - z_0| = R \}$ is the circle centered at z_0 with radius $R > 0$, oriented in the counterclockwise direction, and n is an integer, then

$$\oint_\Gamma \frac{dz}{(z - z_0)^n} = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

Hint: Parametrize the circle by $z = z_0 + Re^{it}, \; 0 \le t \le 2\pi$.

SOLUTION: If $n = 1$ and $z = z_0 + Re^{it}$, then $dz = z'(t)dt = iRe^{it}dt$, and the integral becomes

$$\oint_\Gamma \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}\, dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

If $n \neq 1$ and $z = z_0 + Re^{it}$, then $dz = z'(t)dt = iRe^{it}dt$, and the integral becomes

$$\begin{aligned} \oint_\Gamma \frac{dz}{(z - z_0)^n} &= \int_0^{2\pi} \frac{iRe^{it}}{R^n e^{int}}\, dt = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{-(n-1)it}\, dt \\ &= \frac{-i}{(n-1)iR^{n-1}} e^{-(n-1)it} \Big|_0^{2\pi} = \frac{-1}{(n-1)R^{n-1}} \left[e^{-(n-1)2\pi i} - e^0 \right] \\ &= \frac{-1}{(n-1)R^{n-1}} [1 - 1] = 0. \end{aligned}$$

Question 2. (10 points) Obtain the expansion of the function

$$f(z) = \frac{(z-1)}{z^2}$$

into its *Laurent series*, valid in the domain $0 < |z| < \infty$.

SOLUTION: If we rewrite the function as

$$f(z) = \frac{z}{z^2} - \frac{1}{z^2} = \frac{1}{z} - \frac{1}{z^2},$$

then this is the Laurent series for $f(z)$, valid for $0 < |z| < \infty$.

Question 3. (10 points) For $z \neq 0$, define the function

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Show that $f(z)$ is one-to-one on the domain $D = \{\, z \in \mathbb{C} \,\big|\, |z| > 1 \,\}$; that is, show that if $z \in D$ and $w \in D$ and $f(z) = f(w)$, then $z = w$.

SOLUTION: Suppose that $z, w \in D$ and $f(z) = f(w)$, then

$$z + \frac{1}{z} = w + \frac{1}{w},$$

so that

$$z - w = \frac{1}{w} - \frac{1}{z} = \frac{z - w}{zw}.$$

Therefore,

$$(zw - 1)(z - w) = 0,$$

and if $z \neq w$, then $zw = 1$. However, this implies that $|zw| = |z|\,|w| = 1$, and if $|z| > 1$, then we must have $|w| < 1$, which contradicts the fact that both z and w are in D. So the assumption that $z \neq w$ leads to a contradiction, and we must have $z = w$.

Therefore, if $f(z) = f(w)$ for $z, w \in D$, then $z = w$, and f is one-to-one.

Question 4. (10 points) If $f(z)$ is analytic within and on a simple closed contour Γ, and z_0 is inside Γ, show that

$$\oint_\Gamma \frac{f(z)\,dz}{(z - z_0)^2} = \oint_\Gamma \frac{f'(z)\,dz}{z - z_0}.$$

Hint: Use the Cauchy Integral Formula.

SOLUTION: From the Cauchy Integral Formula applied to $f(z)$ we have

$$f'(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z - z_0)^2}.$$

From the Cauchy Integral Formula applied to $f'(z)$ (which is also analytic), we have

$$f'(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f'(z)\,dz}{z - z_0}.$$

Question 5. (10 points) If $f(z)$ and $g(z)$ are both analytic at z_0, and

$$f(z_0) = g(z_0) = f'(z_0) = g'(z_0) = \cdots = f^{(m-1)}(z_0) = g^{(m-1)}(z_0) = 0,$$

but $g^{(m)}(z_0) \neq 0$, show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}$$

(*Generalized L'Hospital's Rule*).

SOLUTION: The Taylor series for $f(z)$ and $g(z)$ about the point z_0 are given by

$$f(z) = f^{(m)}(z_0)(z - z_0)^m + f^{(m+1)}(z_0)(z - z_0)^{m+1} + \cdots ,$$
$$g(z) = g^{(m)}(z_0)(z - z_0)^m + g^{(m+1)}(z_0)(z - z_0)^{m+1} + \cdots ,$$

and the quotient is given by

$$\frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0) + f^{(m+1)}(z_0)(z - z_0) + \cdots}{g^{(m)}(z_0) + g^{(m+1)}(z_0)(z - z_0) + \cdots}.$$

Letting $z \to z_0$, both series converge, and

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}.$$