MATH 309

MIDTERM EXAMINATION I SOLUTIONS

DATE: Friday June 9, 2017 TIME: 50 Minutes

Question 1. **(10 points)** Determine all values of $z = a + ib$ such that

$$z^2 = i.$$

SOLUTION. We have $z^2 = (a+ib)^2 = a^2 - b^2 + 2iab = i$, and equating real and imaginary parts,

$$a^2 - b^2 = 0 \qquad \text{and} \qquad 2ab = 1.$$

From the first equation, we have $a = b$ or $a = -b$. If $a = -b$, then we have $-2a^2 = -2b^2 = 1$ from the second equation, which is impossible, since a and b are real numbers. Therefore, $a = b$, and from the second equation, we have $a^2 = b^2 = 1/2$. Therefore, the solutions to $z^2 = i$ are

$$z = \pm\frac{1}{\sqrt{2}}\,(1+i)\,.$$

Question 2. **(10 points)** Express the number

$$z = \cos\pi(1+i)$$

in the form $a + ib$, where a and b are real numbers.

SOLUTION. We have

$$\cos\pi(1+i) = \frac{e^{i\pi(1+i)} + e^{-i\pi(1+i)}}{2} = \frac{e^{i\pi}e^{-\pi} + e^{-i\pi}e^{\pi}}{2}$$
$$= \frac{-e^{-\pi} - e^{\pi}}{2} = -\frac{e^{\pi} + e^{-\pi}}{2},$$

and $\cos\pi(1+i) = -\cosh\pi$.

Question 3. **(10 points)** Let $z \in \mathbb{C}$ with $|z| = 1$, where $z \neq 1$. Show that

$$\lim_{n\to\infty} z^n$$

does not exist.

Hint: Assume that $\lim\limits_{n\to\infty} z^n = \gamma$ does exist, and show that this leads to a contradiction.

SOLUTION. Suppose that $|z| = 1$ where $z \neq 1$, and $\lim\limits_{n\to\infty} z^n = \gamma$, then

$$\lim_{n\to\infty} z^{n+1} = z \lim_{n\to\infty} z^n = z\cdot\gamma.$$

However,

$$\lim_{n\to\infty} z^{n+1} = \lim_{m\to\infty} z^m = \gamma$$

also, so that $z\cdot\gamma = \gamma$, that is, $(z-1)\gamma = 0$, and since $z \neq 1$, this implies that $\gamma = 0$. However, this implies that $\lim\limits_{n\to\infty} z^n = 0$, which is impossible, since $|z^n| = |z|^n = 1$ for all $n \geq 1$.

Question 4. (10 points) Let

$$f(z) = \frac{y}{x^2 + y^2} + i\frac{x}{x^2 + y^2}$$

for $z = x + iy \neq 0$. Show that $f'(z)$ exists for all $z \neq 0$ and write $f'(z)$ in terms of z.

Solution. The real and imaginary parts of $f(z)$ are given by

$$u(x, y) = \frac{y}{x^2 + y^2} \qquad \text{and} \qquad v(x, y) = \frac{x}{x^2 + y^2},$$

and taking partial derivatives, we have

$$\frac{\partial u}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}$$

for all $(x, y) \neq (0, 0)$. Since all the partial derivatives exist and are continuous for $(x, y) \neq (0, 0)$ and the Cauchy-Riemann equations hold for all $(x, y) \neq (0, 0)$, then $f'(z)$ exists for each $z \neq 0$. Also,

$$f'(z) = \frac{-2xy}{(x^2 + y^2)^2} + i\,\frac{y^2 - x^2}{(x^2 + y^2)^2} = -i\frac{[x^2 - y^2 - 2ixy]}{(x^2 + y^2)^2} = \frac{-i(\overline{z})^2}{z^2(\overline{z})^2} = -\frac{i}{z^2}.$$

This is not too surprising, since $f(z) = \dfrac{i}{z}$ for all $z \in D$.

Question 5. (10 points) Let

$$f(z) = \ln r + i\theta,$$

where $z = re^{i\theta}$, with $z \in D = \{(r, \theta) \,|\, r > 0,\ -\pi < \theta < \pi\}$. Use the polar form of the Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

to show that $f(z)$ is analytic at each $z \in D$, and write $f'(z)$ in terms of z.

Solution. The function $f(z)$ is continuous at each point $z \in D$, with real and imaginary parts

$$u(r, \theta) = \ln r \qquad \text{and} \qquad v(r, \theta) = \theta.$$

The function $f(z)$ is continuous at each point $z \in D$ and the partial derivatives are continuous on D and satisfy the Cauchy-Riemann equations at each point of D, since

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = \frac{\partial v}{\partial r}.$$

Therefore, from the Cauchy-Riemann Theorem, $f'(z)$ exists for each z in the open connected set D, and so

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

for all $z \in D$. Again, this is not too surprising since $f(z) = \text{Log } z$ for $z \in D$ (the principal value of the complex logarithm function).