Math 309 Spring-Summer 2017

Mathematical Methods for Electrical Engineers

Limits in the Euclidean Plane and in the Complex Plane

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Date: Wednesday, May 24, 2017

Limits in the Complex Plane

In this note we show that limits in the complex plane $\mathbb{C}$ are exactly the same as limits in the plane $\mathbb{R}^2$ equipped with the Euclidean norm, and we prove the following theorem.

Theorem. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$, then

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0.$$

Proof. Suppose that $\lim_{z \to z_0} f(z) = w_0$, then given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon$$

whenever $0 < |z - z_0| < \delta$, and therefore

$$|u(x,y) - u_0| \leq \sqrt{(u(x,y) - u_0)^2 + (v(x,y) - v_0)^2} = |f(z) - w_0| < \epsilon$$

and

$$|v(x,y) - v_0| \leq \sqrt{(u(x,y) - u_0)^2 + (v(x,y) - v_0)^2} = |f(z) - w_0| < \epsilon$$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} = |z - z_0| < \delta$, that is,

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0.$$

Conversely, suppose that

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0 \qquad \text{and} \qquad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0,$$

and let $\epsilon > 0$. Choose $\delta > 0$ so that

$$|u(x,y) - u_0| < \frac{\epsilon}{\sqrt{2}} \qquad \text{and} \qquad |v(x,y) - v_0| < \frac{\epsilon}{\sqrt{2}}$$

whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

If $|z - z_0| < \delta$, then

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = |z - z_0| < \delta,$$

implies that

$$|f(z) - w_0| = \sqrt{(u(x, y) - u_0)^2 + v(x, y) - v_0)^2} < \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} = \epsilon,$$

and therefore

$$\lim_{z \to z_0} f(z) = w_0.$$

□

Limits in the Extended Complex Plane

Recall that if z is a point in the *finite* complex plane $\mathbb{C}$, we said that z **tends to** ∞ if and only if $|z| \to \infty$. Now we want to describe what we mean by

$$\lim_{z \to z_0} f(z) = w_0$$

when z_0 or w_0, or both, is replaced by the point at infinity. This is fairly straightforward, we simply replace the appropriate neighourhoods of z_0 and w_0 by neighbourhoods of infinity.

Theorem. Let z_0 and w_0 be points in the z and w planes, respectively, then we have

$$\lim_{z \to z_0} f(z) = \infty \qquad \text{if and only if} \qquad \lim_{z \to z_0} \frac{1}{f(z)} = 0 \tag{$*$}$$

and

$$\lim_{z \to \infty} f(z) = w_0 \qquad \text{if and only if} \qquad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0. \tag{$**$}$$

Also,

$$\lim_{z \to \infty} f(z) = \infty \qquad \text{if and only if} \qquad \lim_{z \to 0} \frac{1}{f(1/z)} = 0. \tag{$***$}$$

Proof. To prove $(*)$, note that $\lim_{z \to z_0} f(z) = \infty$ if and only if given any $\epsilon > 0$, no matter how small, there exists a $\delta > 0$ such that

$$|f(z)| > \frac{1}{\epsilon} \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta$$

, that is, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{1}{f(z)} - 0 \right| < \epsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta,$$

that is, $\lim_{z \to z_0} \frac{1}{f(z)} = 0$.

To prove $(**)$, note that $\lim_{z \to \infty} f(z) = w_0$ if and only if given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \qquad \text{whenever} \qquad |z| > \frac{1}{\delta},$$

that is, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| f\left(\frac{1}{z}\right) - w_0 \right| < \epsilon \qquad \text{whenever} \qquad 0 < |z - 0| < \delta,$$

that is, $\lim_{z \to 0} f(1/z) = w_0$.

Finally, to prove $(***)$, note that $\lim_{z \to \infty} f(z) = \infty$ if and only if given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z)| > \frac{1}{\epsilon} \qquad \text{whenever} \qquad |z| > \frac{1}{\delta},$$

that is, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{1}{f(1/z)} - 0 \right| < \epsilon \qquad \text{whenever} \qquad 0 < |z - 0| < \delta,$$

(obtained by replacing z by $1/z$), that is, $\lim_{z \to 0} \dfrac{1}{f(1/z)} = 0$.

□

For example,

$$\lim_{z \to 1} \frac{z+1}{z-1} = \infty \qquad \text{since} \qquad \lim_{z \to 1} \frac{z-1}{z+1} = 0,$$

and

$$\lim_{z \to \infty} \frac{2z^2 - 1}{z + 1} = \infty, \qquad \text{since} \qquad \lim_{z \to 0} \frac{(1/z) + 1}{2/z^2 - 1} = \lim_{z \to 0} \frac{z + z^2}{2 - z^2} = 0.$$