



Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers
Jordan's Lemma

Department of Mathematical and Statistical Sciences
University of Alberta

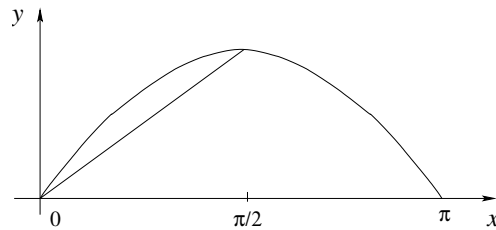
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In this note we will prove *Jordan's Inequality*:

Lemma. If $R > 0$, then

$$\int_0^\pi e^{-R \sin \theta} d\theta \leq \frac{\pi}{R} (1 - e^{-R}).$$

Proof. From the figure



it is clear that

$$0 < \frac{2}{\pi} \theta < \sin \theta$$

for $0 < \theta < \frac{\pi}{2}$.

If $R > 0$, then

$$\frac{2R}{\pi} \theta < R \sin \theta,$$

so that

$$-R \sin \theta < -\frac{2R}{\pi} \theta,$$

and since the exponential function is monotone increasing, we have

$$e^{-R \sin \theta} < e^{-\frac{2R\theta}{\pi}}$$

for $0 < \theta < \frac{\pi}{2}$.

Therefore,

$$\int_0^{\frac{\pi}{2}} e^{-R \sin \theta} d\theta \leq \int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}} d\theta = -\frac{\pi}{2R} e^{-\frac{2R\theta}{\pi}} \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2R} (1 - e^{-R}),$$

and

$$\int_0^{\frac{\pi}{2}} e^{-R \sin \theta} d\theta \leq \frac{\pi}{2R} (1 - e^{-R}). \quad (*)$$

Making the substitution $\phi = \pi - \theta$ in the integral of $e^{-R \sin \theta}$ over the interval $\frac{\pi}{2} < \theta < \pi$, we have

$$\int_{\frac{\pi}{2}}^{\pi} e^{-R \sin \theta} d\theta = - \int_{\frac{\pi}{2}}^0 e^{-R \sin(\pi-\phi)} d\phi = \int_0^{\frac{\pi}{2}} e^{-R \sin \phi} d\phi,$$

so that

$$\int_{\frac{\pi}{2}}^{\pi} e^{-R \sin \theta} d\theta = \int_0^{\frac{\pi}{2}} e^{-R \sin \phi} d\phi,$$

and

$$\int_{\frac{\pi}{2}}^{\pi} e^{-R \sin \theta} d\theta \leq \frac{\pi}{2R} (1 - e^{-R}). \quad (**)$$

From (*) and (**) we have

$$\int_0^{\pi} e^{-R \sin \theta} d\theta = \int_0^{\frac{\pi}{2}} e^{-R \sin \theta} d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-R \sin \theta} d\theta \leq \frac{\pi}{2R} (1 - e^{-R}) + \frac{\pi}{2R} (1 - e^{-R}),$$

so that

$$\int_0^{\pi} e^{-R \sin \theta} d\theta \leq \frac{\pi}{R} (1 - e^{-R}).$$

□

Now we can state:

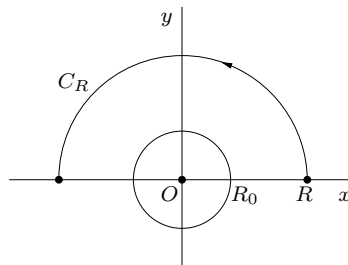
Jordan's Lemma:

- (a) Let $f(z)$ be analytic at all points in the upper half plane $y \geq 0$ that are exterior to a circle $|z| = R_0$.
- (b) Let C_R denote the semicircle $z = R e^{i\theta}$, $0 \leq \theta \leq \pi$, where $R > R_0$.
- (c) Suppose that for all points z on C_R , there is a positive constant M_R such that

$$|f(z)| \leq M_R \quad \text{and} \quad \lim_{R \rightarrow \infty} M_R = 0.$$

Then, for every positive constant a ,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz = 0.$$



Proof. If statements (a), (b), and (c) are true, then using the parametrization of the semicircle, we have

$$\int_{C_R} f(z) e^{iaz} dz = \int_0^{\pi} f(R e^{i\theta}) e^{ia R e^{i\theta}} R i e^{i\theta} d\theta.$$

Now,

$$|f(Re^{i\theta})| \leq M_R \quad \text{and} \quad |e^{iaRe^{i\theta}}| \leq e^{-aR \sin \theta},$$

and from Jordan's inequality we have

$$\left| \int_{C_R} f(z) e^{iaz} dz \right| \leq M_R R \int_0^\pi e^{-aR \sin \theta} d\theta < \frac{M_R \pi}{a} (1 - e^{-aR}),$$

so that

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz = 0.$$

since $M_R \rightarrow 0$ as $R \rightarrow \infty$.

□