



Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers

The Exponential Function as a Limit

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In this note, using a series of lemma, we show that

$$e^z = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n$$

for each $z \in \mathbb{C}$. We assume that the Maclaurin series expansion $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, valid for all $z \in \mathbb{C}$, is known.

Lemma 1. If $0 \leq x_i \leq 1$ for $1 \leq i \leq n$, then

$$1 - \sum_{i=1}^n x_i \leq \prod_{i=1}^n (1 - x_i) \quad (*)$$

Proof. The proof is by mathematical induction.

Base Case: For $n = 1$, if $0 \leq x_1 \leq 1$, then

$$1 - \sum_{i=1}^n x_i = 1 - x_1 = \prod_{i=1}^n (1 - x_i),$$

and $(*)$ holds for $n = 1$.

Inductive Step: Assume that $(*)$ is true for some integer $n \geq 1$, and suppose that $0 \leq x_i \leq 1$ for $1 \leq i \leq n+1$, then

$$\begin{aligned} \prod_{i=1}^{n+1} (1 - x_i) &= (1 - x_{n+1}) \prod_{i=1}^n (1 - x_i) \\ &\geq (1 - x_{n+1}) \left(1 - \sum_{i=1}^n x_i\right) \\ &= 1 - \sum_{i=1}^n x_i - x_{n+1} + x_{n+1} \sum_{i=1}^n x_i \\ &\geq 1 - \sum_{i=1}^{n+1} x_i, \end{aligned}$$

and $(*)$ is true for $n + 1$.

By the principle of mathematical induction, $(*)$ is true for all $n \geq 1$.

□

Lemma 2. For $n \geq 1$ and $z \in \mathbb{C}$, we have

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| \leq \left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right| \leq \frac{|z|^2 e^{|z|}}{2n}.$$

Proof. We have

$$\begin{aligned} \left| e^z - \left(1 + \frac{z}{n}\right)^n \right| &= \left| \sum_{k=0}^{\infty} \frac{z^k}{k!} - \sum_{k=0}^n \binom{n}{k} \frac{z^k}{n^k} \right| && \text{(Maclaurin expansion of } e^z \text{ and binomial theorem)} \\ &= \left| \sum_{k=2}^{\infty} \frac{z^k}{k!} - \sum_{k=2}^n \binom{n}{k} \frac{z^k}{n^k} \right| && \text{(first two terms the same in both sums)} \\ &= \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} + \sum_{k=2}^n \left[1 - \frac{k!}{n^k} \binom{n}{k} \right] \frac{z^k}{k!} \right| \\ &= \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} + \sum_{k=2}^n \underbrace{\left[1 - \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \right]}_{\geq 0} \frac{z^k}{k!} \right| \\ &\leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^n \left[1 - \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \right] \frac{|z|^k}{k!} \\ &= \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^n \left[1 - \frac{k!}{n^k} \binom{n}{k} \right] \frac{|z|^k}{k!} \\ &= \left| \sum_{k=2}^{\infty} \frac{|z|^k}{k!} - \sum_{k=2}^n \binom{n}{k} \frac{|z|^k}{n^k} \right| \\ &= \left| \sum_{k=0}^{\infty} \frac{|z|^k}{k!} - \sum_{k=0}^n \binom{n}{k} \frac{|z|^k}{n^k} \right| \\ &= \left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right|, \end{aligned}$$

therefore

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| \leq \left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right|$$

for all $n \geq 1$ and $z \in \mathbb{C}$.

If we knew that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{|z|}{n}\right)^n = e^{|z|}$$

(which is true for the real exponential function), then we would be done. However, if you haven't seen that before, we continue onward:

In order to prove the second inequality, from the above, we have

$$\begin{aligned}
\left| e^{|z|} - \left(1 + \frac{z}{n}\right)^n \right| &\leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^n \left[1 - \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \right] \frac{|z|^k}{k!} \\
&\leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^n \left(\sum_{i=1}^{k-1} \frac{i}{n} \right) \frac{|z|^k}{k!} \quad (\text{Lemma 1}) \\
&\leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} + \sum_{k=2}^n \frac{k(k-1)}{2n} \frac{|z|^k}{k!} \\
&= |z|^2 \sum_{k=n+1}^{\infty} \frac{|z|^{k-2}}{k!} + \frac{|z|^2}{2n} \sum_{k=2}^n \frac{|z|^{k-2}}{(k-2)!} \\
&= \frac{|z|^2}{n(n+1)} \sum_{\ell=n-1}^{\infty} \frac{n(n+1)|z|^{\ell}}{(\ell+2)!} + \frac{|z|^2}{2n} \sum_{\ell=0}^{n-2} \frac{|z|^{\ell}}{\ell!} \\
&\leq \frac{|z|^2}{2n} \sum_{\ell=n-1}^{\infty} \frac{|z|^{\ell}}{\ell!} + \frac{|z|^2}{2n} \sum_{\ell=0}^{n-2} \frac{|z|^{\ell}}{\ell!} \quad (\text{since } (\ell+2)! \geq n(n+1) \cdot \ell! \text{ for } \ell \geq n-1) \\
&= \frac{|z|^2}{2n} \sum_{\ell=0}^{\infty} \frac{|z|^{\ell}}{\ell!} \\
&= \frac{|z|^2}{2n} e^{|z|}
\end{aligned}$$

that is,

$$\left| e^{|z|} - \left(1 + \frac{|z|}{n}\right)^n \right| \leq \frac{|z|^2}{2n} e^{|z|}$$

for $n \geq 1$ and $z \in \mathbb{C}$.

□

Theorem. If $z \in \mathbb{C}$, then $\lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n = e^z$.

Proof. Let n tend to infinity in the inequality

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| \leq \frac{|z|^2}{2n} e^{|z|}.$$

□