Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers

Differentiation and Integration of Power Series

Taylor's Theorem and Laurent's Theorem

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• Differentiating Power Series Term-by-Term

In this note we will show that a power series $\sum_{n=0}^{\infty} a_n z^n$ can be differentiated or integrated term by term inside its circle of convergence. First we need the following lemma.

Lemma. If $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then

$$\sum_{n=0}^{\infty} n a_n z^{n-1} \qquad \text{and} \qquad \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

also have radius of convergence R.

Proof. Let $z \in \mathbb{C}$ with $|z| < R$ and choose r with $|z| < r < R$, then

$$\left|n a_n z^{n-1}\right| = \left|n a_n \left(\frac{z}{r}\right)^{n-1} \cdot r^{n-1}\right| = \frac{n}{r}\left|\frac{z}{r}\right|^{n-1} \cdot |a_n| r^n,$$

however,

$$\lim_{n\to\infty} n\left|\frac{z}{r}\right|^{n-1} = \lim_{n\to\infty} n e^{(n-1)\ln|z/r|} = 0$$

since $|z| < r$, so there exists an integer n_0 such that

$$\frac{n}{r}\left|\frac{z}{r}\right|^{n-1} < 1$$

for all $n \geq n_0$, and so

$$\left|n a_n z^{n-1}\right| < |a_n| r^n$$

for all $n \geq n_0$.

Now, $\displaystyle\sum_{n=0}^{\infty} |a_n| r^n$ converges since $r < R$, and from the comparison test this implies that $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely for all $z \in \mathbb{C}$ with $|z| < R$.

On the other hand, if $z \in \mathbb{C}$ with $|z| > R$, we can choose a positive number r such that $R < r < |z|$, and if $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges, then $r < |z|$ implies that $\displaystyle\sum_{n=1}^{\infty} n |a_n| r^{n-1}$ converges. However,

$$n|a_n| r^{n-1} \geq \frac{1}{r}|a_n| r^n$$

implies that $\sum_{n=1}^{\infty} |a_n| r^n$ converges, which in turn implies that $\sum_{n=1}^{\infty} a_n r^n$ converges. This is a contradiction, since $r > R$. Therefore, $\sum_{n=1}^{\infty} n a_n z^{n-1}$ diverges for all $z \in \mathbb{C}$ with $|z| > R$.

We have shown that $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges for $|z| < R$, and diverges for $|z| > R$, that is, the radius of convergence of this power series is also R.

Now let $R' > 0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$, then from the above, the series $\sum_{n=0}^{\infty} a_n z^n$ also has radius of convergence R', and therefore $R' = R$. $\square$

And now the promised result, with nary a mention of uniform convergence.

Theorem. If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < R$, is differentiable (and therefore continuous) for $|z| < R$; and

(a) $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ for $|z| < R$,

(b) $\int_0^z f(s)\, ds = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$ for any path C joining 0 and z which lies entirely inside the circle of convergence.

Proof.

(a) Let $z \in \mathbb{C}$ with $|z| < R$, and choose $H > 0$ so that $|z| + H < R$ (z and H are **fixed**).
Now let $h \in \mathbb{C}$ be such that $0 < |h| \leq H$ and define

$$f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

for $|z| < R$, then

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} - f_1(z) &= \sum_{n=0}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n}{h} \right\} - \sum_{n=1}^{\infty} n z^{n-1} \\
&= \sum_{n=1}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n}{h} \right\} - \sum_{n=1}^{\infty} n z^{n-1} \\
&= \sum_{n=1}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n - n z^{n-1} h}{h} \right\} \\
&= \sum_{n=2}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n - n z^{n-1} h}{h} \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| &= \left|\sum_{n=2}^{\infty} a_n \left\{\frac{(z+h)^n - z^n - nz^{n-1}h}{h}\right\}\right| \\
&= \left|\sum_{n=2}^{\infty} a_n \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-1}\right| \\
&\leq |h| \cdot \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} |h|^{k-2} \\
&\leq |h| \cdot \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} H^{k-2} \\
&= \frac{|h|}{H^2} \cdot \sum_{n=2}^{\infty} |a_n| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} H^{k} \\
&\leq \frac{|h|}{H^2} \cdot \sum_{n=1}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z|^{n-k} H^{k} \\
&= \frac{|h|}{H^2} \cdot \sum_{n=1}^{\infty} |a_n| (|z|+H)^n \\
&\leq \frac{|h|}{H^2} \cdot M
\end{aligned}$$

where $M = \sum\limits_{n=1}^{\infty} |a_n|(|z|+H)^n < \infty$ since $|z| + H < R$.

Therefore, we have

$$\left|\frac{f(z+h)-f(z)}{h} - f_1(z)\right| \leq \frac{|h|}{H^2} \cdot M$$

for $|h| \leq H$. Letting $h \to 0$, then

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f_1(z)$$

for $|z| < R$, that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

for $|z| < R$.

(b) Now let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$, and define

$$F(z) = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

for $|z| < R$, then from part (a), f is analytic in the domain $|z| < R$, and so is continuous there, and

$$F'(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $|z| < R$, that is, F is an antiderivative of f in the domain $|z| < R$.

Therefore, if $|z| < R$ and C is any contour joining 0 and z which lies entirely inside the circle of convergence, then

$$\int_C f(s)\, ds = F(z) - F(0) = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

for $|z| < R$.

$\square$

Note: If the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$, then $f'(z)$ exists for all z with $|z| < R$, that is, **any power series is an analytic function inside its circle of convergence**.

• Taylor Series

We proved **Taylor's Theorem** earlier in class, and prove it here again in case you missed it.

Theorem. If $f(z)$ is analytic inside the disk $\Gamma = \{\, z \in \mathbb{C} \,|\, |z - z_0| < R \,\}$, centered at z_0 with radius R, then $f(z)$ has a unique power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for $n = 0, 1, 2, \ldots$.

Proof. We will give the proof for the Maclaurin series of the function $g(z) = f(z + z_0)$, and then you can replace z by $z - z_0$ in this Maclaurin series for the general statement.

Suppose that $g(z)$ is analytic inside the disk Γ of radius R centered at the origin. Write $|z| = r$ and let Γ_0 denote any positively oriented circle $|z| = r_0$, where $r < r_0 < R$, then from the Cauchy Integral Formula we have

$$g(z) = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{g(s)\, ds}{s - z}.$$

Since

$$\frac{1}{s - z} = \frac{1}{s} \cdot \frac{1}{1 - \frac{z}{s}},$$

for $z \neq 1$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z},$$

and replacing z by z/s, we get

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}.$$

Multiplying this by $g(s)$ and integrating with respect to s around Γ_0, we have

$$\int_{\Gamma_0} \frac{g(s)\,ds}{s-z} = \sum_{n=0}^{N-1} \int_{\Gamma_0} \frac{g(s)\,ds}{s^{n+1}}\, z^N + z^N \int_{\Gamma_0} \frac{g(s)\,ds}{(s-z)s^N}.$$

From the generalized Cauchy Integral Formula, we can write this as

$$g(z) = \sum_{n=0}^{N-1} \frac{g^{(n)}(0)}{n!}\, z^n + \rho_N(z),$$

where the remainder is given by

$$\rho_N(z) = \frac{z^N}{2\pi i} \int_{\Gamma_0} \frac{g(s)\,ds}{(s-z)s^N}.$$

To show that this remainder goes to 0 as $N \to \infty$, note that since $|z| = r$ and Γ_0 has radius $r_0 > r$, if s is a point on Γ_0, then

$$|s-z| \geq \big|\,|s| - |z|\,\big| = r_0 - r,$$

and

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0^N} \cdot 2\pi r_0 = \frac{M r_0}{r_0 - r} \left(\frac{r}{r_0}\right)^N,$$

where M is the maximum value of $|g(s)|$ for s on Γ_0. And since $r/r_0 < 1$, then $\rho_N \to 0$ as $N \to \infty$.

Therefore, the Maclaurin series for $g(z) = f(z + z_0)$ converges, and we have

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}\, z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\, z^n,$$

for $|z| < R$. Replacing z by $z - z_0$ in this series, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z - z_0)^n$$

for $|z - z_0| < R$.

$\square$

• Examples

(1). If $f(z) = e^z$, then f is an entire function, and since $f^{(n)}(0) = e^0 = 1$ for all $n \geq 0$, the Maclaurin series for e^z is (no surprise)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots ,$$

that is, the same series we had for real-valued functions. Since $f(z) = e^z$ is an entire function, the Maclaurin series converges for all $z \in \mathbb{C}$.

(2). If $f(z) = \sin z$, then f is an entire function, and we can use the previous example to calculate its Maclaurin series. We have

$$\sin z = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}\left[1 - (-1)^n\right]\frac{i^n z^n}{n!},$$

and only the terms with n odd survive. However, $i^{2n+1} = (i^2)^n i = (-1)^n i$, so that

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

valid for all $z \in \mathbb{C}$.

(3). If $f(z) = \cos z$, then f is an entire function, and since it can be differentiated term-by-term inside its circle of convergence (in this case $\mathbb{C}$), we have

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{d}{dz}\left(z^{2n+1}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

valid for all $z \in \mathbb{C}$.

(4). If $f(z) = \dfrac{1}{1-z}$, then f is analytic everywhere except at $z = 1$, and since

$$f^{(n)}(0) = n!,$$

for $n \geq 0$, then the Maclaurin series for f is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots ,$$

valid for $|z| < 1$.

(5). If $f(z) = \dfrac{1}{1+z}$, then the Maclaurin series for f is

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \cdots ,$$

valid for $|z| = |-z| < 1$.

(6). If we want to expand $f(z) = \dfrac{1+2z^2}{z^3+z^5}$, into a series involving powers of z, we have

$$f(z) = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right),$$

and we cannot find a Maclaurin series for $f(z)$ since it is not analytic at $z = 0$. However, since

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 - \cdots$$

for $|z| < 1$. We do have an expansion of the form

$$f(z) = \frac{1}{z^3}\left(2 - 1 + z^2 - z^4 + z^6 - z^8 + \cdots\right) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots ,$$

valid for $0 < |z| < 1$. This expansion involving negative powers of $z - z_0$ is called a **Laurent series**.

• Laurent Series

We will not prove **Laurent's Theorem** in class, but will provide the proof here in case you want to see it. You are, however, responsible for being able to use it.

Theorem. Let $f(z)$ be analytic in the domain $D = \{\, z \in \mathbb{C} \,|\, r_1 < |z - z_0| < r_2 \,\}$, then for z in this annulus, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s - z_0)^{n+1}}$$

for $n = 0, 1, 2, \ldots$, and

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{(s - z_0)^{-n+1}}$$

for $n = 1, 2, \ldots$ and C is a positively oriented simple closed contour around z_0 lying entirely in D.

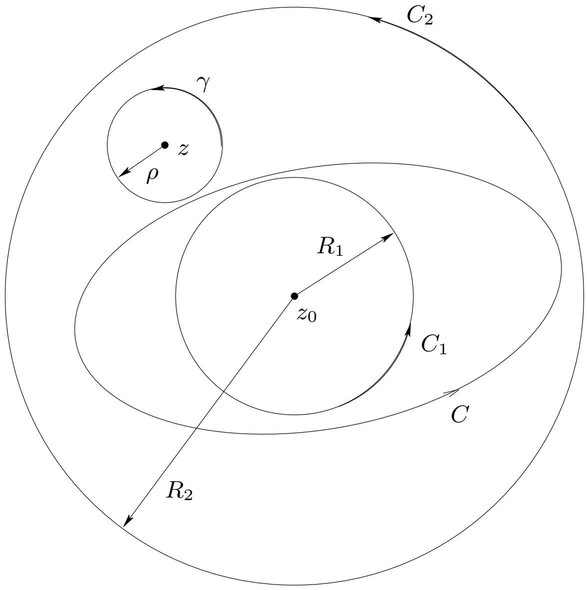
Proof. In the figure below, we let z be a point in the annulus D, and choose R_1 and R_2 such that

$$r_1 < R_1 < |z - z_0| < R_2 < r_2,$$

and let C_1 be the circle centered at z_0 with radius R_1 and C_2 be the circle centered at z_0 with radius R_2. Also, let γ be a circle centered at z such that γ is entirely contained in the annulus D:

$$\gamma = \{\, \zeta \in \mathbb{C} \,|\, |\zeta - z| = \rho \,\}$$

where $\rho > 0$. All of the circles are positively oriented.



From the Cauchy-Goursat Theorem for multiply-connected domains and the Cauchy Integral Formula, we have

$$\oint_{C_2} \frac{f(s)\,ds}{s - z} - \oint_{C_1} \frac{f(s)\,ds}{s - z} - \underbrace{\oint_{\gamma} \frac{f(s)\,ds}{s - z}}_{2\pi i f(z)} = 0,$$

and therefore

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(s)\,ds}{s - z} + \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)\,ds}{z - s}. \qquad (*)$$

- Now for the integral over C_2 we have:

$$\frac{1}{s-z} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(s-z_0)^{n+1}},$$

and the series converges using the same argument we used for Taylor's Theorem, since $|z-z_0| < |s-z_0|$. Therefore,

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(s)\,ds}{s-z} = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i}\oint_{C_2} \frac{f(s)\,ds}{(s-z_0)^{n+1}}\right](z-z_0)^n,$$

that is,

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(s)\,ds}{s-z} = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

- For the integral over C_1 we have:

$$\frac{1}{z-s} = \frac{1}{z-z_0-(s-z_0)} = \frac{1}{z-z_0}\cdot\frac{1}{1-\left(\frac{s-z_0}{z-z_0}\right)}$$

and

$$\frac{1}{z-s} = \frac{1}{z-z_0}\sum_{n=0}^{\infty}\frac{(s-z_0)^n}{(z-z_0)^n} = \sum_{n=0}^{\infty}\frac{(s-z_0)^n}{(z-z_0)^{n+1}},$$

and the series converges since $|s-z_0| < |z-z_0|$ on C_1. So that,

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)\,ds}{z-s} &= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)\,ds}{(s-z_0)^{-n}}(z-z_0)^{-n-1}\\ &= \sum_{m=1}^{\infty}\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)\,ds}{(s-z_0)^{-m+1}}(z-z_0)^{-m}, \quad (\text{let } m=n+1)\end{aligned}$$

so that

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)\,ds}{z-s} = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}.$$

From $(*)$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i}\oint_{C_2}\frac{f(s)\,ds}{(s-z)^{n+1}}$$

for $n = 0, 1, 2, \ldots$, and

$$b_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)\,ds}{(s-z)^{-n+1}}$$

for $n = 1, 2, \ldots$.

Since $f(z)$ is analytic in the annular region $r_1 < |z-z_0| < r_2$, from the deformation of path principle we can replace C_1 and C_2 by the contour C $\square$

Note: Taylor series and Laurent series are unique, so if you find an expansion, you have found the Taylor series or Laurent series!