Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers

Differentiation of a Complex Function of a Complex Variable

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In this note we will give necessary and sufficient conditions for a function $f(z)$ of a complex variable $z = x+iy$ to be differentiable at a point $z_0 = x_0 + iy_0$.

First we need the definition of what it means for a real valued function $u(x, y)$ of two real variables x and y to be differentiable at a point (x_0, y_0).

Definition. Let u be a real valued function of two real variables and suppose that $u(x, y)$ is defined on an open neighborhood of a point (x_0, y_0), then u is **differentiable** or **stongly differentiable** at (x_0, y_0) if and only if there exist real numbers A and B such that

$$u(x_0 + h, y_0 + k) = u(x_0, y_0) + A \cdot h + B \cdot k + \varphi(h, k) \cdot \sqrt{h^2 + k^2},$$

where $\varphi(0, 0) = 0$, and $\varphi(h, k) \to 0$ as $\sqrt{h^2 + k^2} \to 0$ (here A and B are independent of h and k).

This is equivalent to saying that

$$\lim_{(h,k)\to(0,0)} \left| \frac{u(x_0 + h, y_0 + k) - u(x_0, y_0) - A \cdot h - B \cdot k}{\sqrt{h^2 + k^2}} \right| = 0. \qquad (*)$$

The linear map $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g(h, k) = A \cdot h + B \cdot k,$$

denoted by

$$g = u'(x_0, y_0), \qquad g = Du(x_0, y_0), \qquad \text{or} \qquad g = \nabla u(x_0, y_0)$$

is called the **derivative** or **Frechét derivative** of u at (x_0, y_0).

Note: If $u(x, y)$ is differentiable at the point (x_0, y_0), then the first partial derivatives with respect to x and y both exist and are given by

$$\frac{\partial u}{\partial x}(x_0, y_0) = A \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_0, y_0) = B.$$

These results follow immediately from the definition of the derivative by first taking $k = 0$ and $h \neq 0$ in $(*)$ and letting $h \to 0$, and then taking $h = 0$ and $k \neq 0$ in $(*)$ and letting $k \to 0$. Therefore, the derivative of u at (x_0, y_0) is given by

$$u'(x_0, y_0)(h, k) = \frac{\partial u}{\partial x}(x_0, y_0) \cdot h + \frac{\partial u}{\partial y}(x_0, y_0) \cdot k$$

for $(h, k) \in \mathbb{R}^2$.

We have the usual theorem concerning continuity of differentiable functions.

Theorem. If u is differentiable at (x_0, y_0), then u is continuous at (x_0, y_0).

Proof. Note that

$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = \left(\frac{u(x_0 + h, y_0 + k) - u(x_0, y_0) - A \cdot h - B \cdot k}{\sqrt{h^2 + k^2}}\right) \cdot \sqrt{h^2 + k^2} + A \cdot h + B \cdot k,$$

and letting $(h, k) \to (0, 0)$,

$$\lim_{(h,k)\to(0,0)} u(x_0 + h, y_0 + k) - u(x_0, y_0) = 0,$$

so that

$$\lim_{(h,k)\to(0,0)} u(x_0 + h, y_0 + k) = u(x_0, y_0),$$

and u is continuous at $x_0, y_0)$.

$\square$

Now we give a sufficient condition for a function u of two real variables to be differentiable at a point.

Theorem. If the real valued function u is defined on an open neighborhood of the point (x_0, y_0), and one of the first partial derivatives exists at each point of the neighborhood and is continuous at (x_0, y_0), while the other first partial derivative exists at (x_0, y_0), then u is differentiable at (x_0, y_0).

Proof. We assume that u_x exists at each point of an open neighborhood of (x_0, y_0) and is continuous at (x_0, y_0), while u_y exists at (x_0, y_0).

We will show that

$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = u_x(x_0, y_0) \cdot h + u_y(x_0, y_0) \cdot k + \varphi(h, k) \cdot (|h| + |k|)$$

where $\varphi(h, k) \to 0$ as $(h, k) \to 0$.

We write

$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = \underbrace{u(x_0 + h, y_0 + k) - u(x_0, y_0 + k)}_{(*)} + \underbrace{u(x_0, y_0 + k) - u(x_0, y_0)}_{(**)}$$

- For the expression $(*)$, the mean value theorem guarantees a point on the line between x_0 and $x_0 + h$, say $x_0 + \theta h$ where $0 < \theta < 1$, such that

  $$u(x_0 + h, y_0 + k) - u(x_0, y_0 + k) = u_x(x_0 + \theta h, y_0 + k) \cdot h,$$

  and since u_x is continuous at (x_0, y_0), then

  $$u_x(x_0 + \theta h, y_0 + k) = u_x(x_0, y_0) + \varphi_1(h, k)$$

  where $\varphi_1(h, k) \to 0$ as $h \to 0$ and $k \to 0$. Therefore,

  $$u(x_0 + h, y_0 + k) - u(x_0, y_0 + k) = u_x(x_0, y_0) \cdot h + \varphi_1(h, k) \cdot h \tag{$*$}$$

  where $\varphi_1(h, k) \to 0$ as $h \to 0$ and $k \to 0$.

- For the expression $(**)$, since $u_y(x_0, y_0)$ exists, we can write

$$\frac{u(x_0, y_0 + k) - u(x_0, y_0)}{k} = u_y(x_0, y_0) + \varphi_2(k)$$

where $\varphi_2(k) \to 0$ as $k \to 0$. If we define $\varphi_2(0) = 0$, then

$$u(x_0, y_0 + k) - u(x_0, y_0) = u_y(x_0, y_0) \cdot k + \varphi_2(k) \cdot k \tag{$**$}$$

where $\varphi_2(k) \to 0$ as $k \to 0$.

Therefore,

$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = u_x(x_0, y_0) \cdot h + u_y(x_0, y_0) \cdot k + \varphi_1(h, k) \cdot h + \varphi_2(k) \cdot k,$$

where $\varphi_1(h, k) \to 0$ and $\varphi_2(k) \to 0$ as $(h, k) \to (0, 0)$.

Now define

$$\varphi(h, k) = \begin{cases} \dfrac{\varphi_1(h, k) \cdot h + \varphi_2(k) \cdot k}{|h| + |k|} & \text{if } (h, k) \neq (0, 0), \\ 0 & \text{if } (h, k) = (0, 0), \end{cases}$$

then for $(h, k) \neq (0, 0)$, we have

$$|\varphi(h, k)| \leq \frac{|\varphi_1(h, k)| \cdot |h|}{|h| + |k|} + \frac{\varphi_2(k)| \cdot |k|}{|h| + |k|} \leq |\varphi_1(h, k)| + |\varphi_2(k)|,$$

so that $\varphi(h, k) \to 0$ as $(h, k) \to (0, 0)$, and thus $u(x, y)$ is differentiable at (x_0, y_0).

$\square$

Finally, we are in a position to give a necessary and sufficient condition for a function $f(z)$ of a complex variable $z = x + iy$ to be differentiable at a point $z_0 = x_0 + iy_0$.

Theorem. (Cauchy-Riemann)

Let f be defined on an open set containing z_0, then $f'(z_0)$ exists if and only if $f(z) = u(x, y) + iv(x, y)$ where u and v are strongly differentiable at (x_0, y_0) and satisfy the **Cauchy-Riemann equations** at (x_0, y_0)

$$u_x(x_0, y_0) = v_y(x_0, y_0) \qquad \text{and} \qquad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Proof. We have to show both implications.

(i) If $f'(z_0)$ exists, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists, and by looking at real and imaginary parts of $f(z)$, this implies that $u(x, y)$ and $v(x, y)$ are strongly differentiable at (x_0, y_0).

If we take $z = x + iy_0$, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0},$$

and letting $x \to x_0$, both the real and imaginary parts of this difference quotient converge to a limit, and therefore

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0). \tag{1}$$

If we take $z = x_0 + iy$, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i\frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)},$$

and letting $y \to y_0$, both the real and imaginary parts of this difference quotient converge to a limit, and therefore

$$f'(z_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0). \tag{2}$$

Equating the real and imaginary parts of (1) and (2), we get the Cauchy-Riemann equations

$$u_x(x_0, y_0) = v_y(x_0, y_0) \qquad \text{and} \qquad u_y(x_0, y_0) = -v_x(x_0, y_0),$$

together with two formulas for $f'(z_0)$

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \qquad \text{and} \qquad f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

(ii). If u and v are differentiable at the point (x_0, y_0), then there exist functions $\varphi_1(x, y)$ and $\varphi_2(x, y)$ such that

$$u(x, y) = u(x_0, y_0) + u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0) + \varphi_1(x, y) \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

where $\varphi_1(x, y) \to 0$ as $(x, y) \to (x_0, y_0)$, and

$$v(x, y) = v(x_0, y_0) + v_x(x_0, y_0)(x - x_0) + v_y(x_0, y_0)(y - y_0) + \varphi_2(x, y) \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

where $\varphi_2(x, y) \to 0$ as $(x, y) \to (x_0, y_0)$.

Now, $f(z) = u(x, y) + iv(x, y)$, so that

$$\begin{aligned} f(z) = u(x_0, y_0) &+ u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0) + \varphi_1(x, y) \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2} \\ &+ iv(x_0, y_0) + iv_x(x_0, y_0)(x - x_0) + iv_y(x_0, y_0)(y - y_0) + \varphi_2(x, y) \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}, \end{aligned}$$

that is,

$$\begin{aligned} f(z) = f(z_0) + [u_x(x_0, y_0) + iv_x(x_0, y_0)]\,(x - x_0) + [u_y(x_0, y_0) + iv_y(x_0, y_0)]\,(y - y_0) \\ + (\varphi_1(x, y) + \varphi_2(x, y)) \cdot |z - z_0|. \end{aligned}$$

Using the Cauchy-Riemann equations, the difference quotient can be written as

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} = u_x(x_0, y_0)\frac{[(x - x_0) + i(y - y_0)]}{z - z_0} &+ iv_x(x_0, y_0)\frac{[(x - x_0) + i(y - y_0)]}{z - z_0} \\ &+ (\varphi_1(x, y) + \varphi_2(x, y)) \cdot \frac{|z - z_0|}{z - z_0}, \end{aligned}$$

that is,

$$\frac{f(z) - f(z_0)}{z - z_0} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\varphi_1(x, y) + \varphi_2(x, y)) \cdot \frac{|z - z_0|}{z - z_0}.$$

Now, as $z \to z_0$, since $\left|\frac{|z - z_0|}{z - z_0}\right| = 1$ and $\varphi_1(x, y) \to 0$ and $\varphi_2(x, y) \to 0$, then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and equals $u_x(x_0, y_0) + iv_x(x_0, y_0)$.

$\square$

The following sufficient conditions for differentiability follow directly from the Cauchy-Riemann Theorem:

Theorem.

If the function $f(z) = u(x, y) + iv(x, y)$ is defined on an ϵ-neighborhood of a point $z_0 = x_0 + iy_0$, and

(a) The first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in this ϵ-neighborhood;

(b) The first-order partial derivatives of u and v are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

at the point (x_0, y_0),

then $f'(z_0)$ exists, and is given by $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.