Department of Mathematical and Statistical Sciences
University of Alberta

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Theorem. If θ is a real number, and n is an integer, then

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \tag{$*$}$$

Proof. The theorem is true for $n = 0$ or $n = 1$: for $n = 0$, $(*)$ becomes

$$\cos 0 + i\sin 0 = (\cos\theta + i\sin\theta)^0,$$

and both sides are 1, since $\cos 0 = 1$ and if $z \neq 0$, $z^0 = 1$ by convention, since not both $\cos\theta$ and $\sin\theta$ are 0. While for $n = 1$, $(*)$ becomes

$$\cos\theta + i\sin\theta = (\cos\theta + i\sin\theta)^1,$$

which is an identity.

Assuming that $(*)$ is true for some $n > 1$, then

$$\begin{aligned}
\cos(n+1)\theta + i\sin(n+1)\theta &= \cos n\theta\cos\theta - \sin\, n\theta\sin\theta + i(\sin\, n\theta\cos\theta + \sin\theta\cos n\theta) \\
&= \cos n\theta(\cos\theta + i\sin\theta) + i\sin\, n\theta(\cos\theta + i\sin\theta) \\
&= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \qquad \text{(by the induction hypothesis)} \\
&= (\cos\theta + i\sin\theta)^{n+1}.
\end{aligned}$$

Therefore, by induction, $(*)$ is true for all integers $n \geq 0$.

If $n < 0$, then $|n| > 0$, and from the preceding argument we have

$$\cos|n|\theta + i\sin|n|\theta = (\cos\theta + i\sin\theta)^{|n|},$$

so that

$$\cos(-n\theta) + i\sin(-n\theta) = \frac{1}{(\cos\theta + i\sin\theta)^n},$$

that is,

$$\cos n\theta - i\sin\, n\theta = \frac{1}{(\cos\theta + i\sin\theta)^n}.$$

Therefore,

$$\frac{1}{\cos n\theta + i\sin\, n\theta} = \frac{1}{(\cos\theta + i\sin\theta)^n},$$

that is,

$$\cos n\theta + i\sin\, n\theta = (\cos\theta + i\sin\theta)^n,$$

and $(*)$ is true for all integers $n \in \mathbb{Z}$. $\square$

Note: We have now shown that $e^{n\theta} = (e^{\theta})^n$