Math 309 Spring-Summer 2017

Mathematical Methods for Electrical Engineers

The Field of Complex Numbers: $\mathbb{C}$

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- If $\mathbb{C}$ is the set of all complex numbers:

$$\mathbb{C} = \{\, z \,|\, z = x + iy, \text{ where } x, y \in \mathbb{R} \,\}$$

with addition and multiplication defined as follows

$$x_1 + iy_1 + x_2 + iy_2 = x_1 + x_2 + i(y_1 + y_2)$$
$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

for $x_1 + iy_1,\ x_2 + iy_2 \in \mathbb{C}$, then $\mathbb{C}$ is a field, that is, $\mathbb{C}$ together with these operations of addition and multiplication satisfies the following axioms:

a_1 : $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in \mathbb{C}$ (*commutative law*)

a_2 : $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$ (*associative law*)

a_3 : there exists an element 0 in $\mathbb{C}$ such that $z + 0 = z = 0 + z$ for all $z \in \mathbb{C}$ (*additive identity*)

a_4 : for each $z \in \mathbb{C}$, there exists a $w \in \mathbb{C}$ such that $z + w = 0 = w + z$ (*additive inverse*)

a_5 : $z_1 \cdot z_2 = z_2 \cdot z_1$ for all $z_1, z_2 \in \mathbb{C}$ (*commutative law*)

a_6 : $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$ (*associative law*)

a_7 : $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3,\ (z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$ (*distributive laws*).

a_8 : there exists an element $1 \in \mathbb{C}$ such that $z \cdot 1 = z = 1 \cdot z$ for all $z \in \mathbb{C}$ (*multiplicative identity*)

a_9 : $1 \neq 0$

a_{10} : for each $z \in \mathbb{C}$ with $z \neq 0$, there exists a $w \in \mathbb{C}$ such that $z \cdot w = 1 = w \cdot z$ (*multiplicative inverse*)

You should verify each of these axioms.

For example, verify that the additive identity in $\mathbb{C}$ is $0 = 0 + i0$, and that the multiplicative identity in $\mathbb{C}$ is $1 = 1 + i0$.

- Note that if $i \in \mathbb{C}$ is the complex number $i = 0 + i$, then

$$i^2 = (0 + i) \cdot (0 + i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1 + i0 = -(1 + i0) = -1.$$

Therefore, we can manipulate expressions involving complex numbers $z = x + iy$ as usual (that is, as if the numbers were real), and replace i^2 by -1 whenever it occurs.

- **Exercise.** Show that we can also define $\mathbb{C}$ as the set of all 2×2 matrices with real entries of the form

  $$z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

  that is,

  $$\mathbb{C} = \left\{ z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

  with the usual definition of matrix addition and matrix multiplictation.

  Here you have to show first that $\mathbb{C}$ is closed under addition and multiplication. Note that

  $$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix},$$

  while

  $$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}.$$

  Clearly, the additive and multiplicative identities are

  $$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

  respectively, while

  $$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$