



Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineers
The Binomial Theorem

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Binomial Theorem. If a and b are complex numbers and n is a positive integer, then

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}. \quad (*)$$

We will prove this using the principle of mathematical induction, but first we give a definition of the binomial coefficients, and a lemma.

Definition. If n and k are nonnegative integers, the **binomial coefficient** $\binom{n}{k}$ is defined to be

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{for } 0 \leq k \leq n \\ 0 & \text{for } k > n, \end{cases}$$

where by convention $0! = 1$.

Pascal's Lemma. If n and k are integers with $1 \leq k \leq n$, then

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof. Let n and k be integers with $1 \leq k \leq n$, then

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k+1)!} [k + n + 1 - k] = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.$$

□

Proof of Binomial Theorem. We assume first that both a and b are nonzero. In this case, $a^0 = 1$ and $b^0 = 1$.

Base case: If $n = 1$, then

$$(a + b)^1 = a + b = \binom{1}{0} a^0 b^{1-0} + \binom{1}{1} a^1 b^{1-1} = \sum_{k=0}^1 \binom{1}{k} a^k b^{1-k}$$

and $(*)$ is true for $n = 1$.

Inductive Step: If $n \geq 1$ and $(*)$ is true for n , then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b) \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} \\
&= \binom{n}{n} a^{n+1} + \underbrace{\sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k}}_{\ell=k+1} + \sum_{k=1}^n \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} b^{n+1} \\
&= \binom{n+1}{n+1} a^{n+1} + \sum_{\ell=1}^n \binom{n}{\ell-1} a^\ell b^{n+1-\ell} + \sum_{k=1}^n \binom{n}{k} a^k b^{n+1-k} + \binom{n+1}{0} b^{n+1} \\
&= \binom{n+1}{n+1} a^{n+1} + \sum_{k=1}^n \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=1}^n \binom{n}{k} a^k b^{n+1-k} + \binom{n+1}{0} b^{n+1} \\
&= \binom{n+1}{n+1} a^{n+1} + \sum_{k=1}^n \underbrace{\left[\binom{n}{k-1} + \binom{n}{k} \right]}_{\text{Pascal's Lemma}} a^k b^{n+1-k} + \binom{n+1}{0} b^{n+1} \\
&= \binom{n+1}{n+1} a^{n+1} + \sum_{k=1}^n \binom{n+1}{k} a^k b^{n+1-k} + \binom{n+1}{0} b^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k},
\end{aligned}$$

and $(*)$ is true for $n+1$ also.

Therefore, by the principle of mathematical induction, $(*)$ is true for all $n \geq 1$.

In the case that one (or both) of a or b is zero, then $a+b=b$ or $a+b=a$, and the sum is either b^n or a^n , and the theorem is true in this case also, provided we interpret the term 0^0 in the sum as 1. \square

Example. If we take $a = -1$ and $b = 1$ in the binomial theorem, then

$$0 = (-1+1)^n = \sum_{k=0}^n (-1)^k \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k}.$$

In the first sum the index k runs through all the even integers from 0 to n , and the last term in the sum occurs when $2k \leq n < 2k+1$, that is, when $k = \left\lfloor \frac{n}{2} \right\rfloor$, that is, when k is the greatest integer less than or equal to $\frac{n}{2}$.

In the second sum the index k runs through all the odd integers from 1 to n , and the last term in the sum occurs when $2k-1 \leq n < 2k$, that is, when $k = \left\lfloor \frac{n+1}{2} \right\rfloor$, that is, when k is the greatest integer less than or equal to $\frac{n+1}{2}$. Therefore,

$$0 = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} - \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n}{2k-1},$$

and
$$\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} = \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n}{2k-1}.$$

Exercise. Prove the identity

$$2^{\frac{n}{2}} \sin \frac{n\pi}{4} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1}$$

for $n \geq 1$.

SOLUTION: From the binomial theorem we have

$$(1+i)^n = \sum_{k=0}^n \binom{n}{k} i^k$$

and

$$(1-i)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k i^k,$$

so that

$$(1+i)^n - (1-i)^n = \sum_{k=0}^n \binom{n}{k} i^k [1 - (-1)^k],$$

and the only terms that survive are the terms when k is odd.

Now if $k = 2m+1$, then

$$i^{2m+1} = i \cdot i^{2m} = i(-1)^m,$$

and since we want $2m+1 \leq n$, then we must have $m \leq \lfloor \frac{n-1}{2} \rfloor$, and

$$(1+i)^n - (1-i)^n = 2i \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2m+1} (-1)^m,$$

and therefore,

$$\frac{(1+i)^n - (1-i)^n}{2i} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k$$

for $n \geq 0$.

Also,

$$\begin{aligned} \frac{1}{2i} [(1+i)^n - (1-i)^n] &= \frac{1}{2i} \left[\left(\frac{1+i}{\sqrt{2}} \right)^n - \left(\frac{1-i}{\sqrt{2}} \right)^n \right] \cdot 2^{\frac{n}{2}} \\ &= \frac{1}{2i} \left[e^{\frac{in\pi}{4}} - e^{-\frac{in\pi}{4}} \right] \cdot 2^{\frac{n}{2}} \\ &= 2^{\frac{n}{2}} \sin \frac{n\pi}{4}, \end{aligned}$$

and therefore,

$$2^{\frac{n}{2}} \sin \frac{n\pi}{4} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1}$$

for $n \geq 0$.

Note: We have used Euler's formulas

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

to write

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

in the above.