Math 309 Spring-Summer 2017
Mathematical Methods for Electrical Engineering

Absolute Value of a Complex Integral

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In this note we will show that if we have a complex valued function of one real variable, then the absolute value of the integral is less than or equal to the integral of the absolute value of the function.

Theorem. Given a complex valued function

$$w(t) = u(t) + iv(t)$$

where $u(t)$ and $v(t)$ are real valued integrable functions over the interval $[a, b]$, then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt.$$

Proof. We have

$$\begin{aligned}
\left|\int_a^b w(t)\,dt\right|^2 &= \int_a^b w(t)\,dt \cdot \overline{\int_a^b w(s)\,ds} \\
&= \int_a^b w(t)\,dt \cdot \int_a^b \overline{w(s)}\,ds \\
&= \int_a^b \int_a^b w(t)\,\overline{w(s)}\,dt\,ds,
\end{aligned}$$

that is,

$$\left|\int_a^b w(t)\,dt\right|^2 = \int_a^b \int_a^b w(t)\,\overline{w(s)}\,dt\,ds.$$

Note that the expression on the left is real, while the integrand on the right does not appear at first glance to be real. However, as we will see, it is real.

If $w(t) = u(t) + iv(t)$, and $w(s) = u(s) + iv(s)$, then we have

$$\begin{aligned}
\int_a^b \int_a^b w(t)\,\overline{w(s)}\,dt\,ds &= \int_a^b \int_a^b \left[u(t)u(s) + v(t)v(s) + i(u(s)v(t) - u(t)v(s))\right]\,dt\,ds \\
&= \int_a^b \int_a^b (u(t)u(s) + v(t)v(s))\,dt\,ds + i\int_a^b \int_a^b (u(s)v(t) - u(t)v(s))\,dt\,ds \\
&= \int_a^b \int_a^b (u(t)u(s) + v(t)v(s))\,dt\,ds + i\underbrace{\left[\int_a^b u(s)\,ds \cdot \int_a^b v(t)\,dt - \int_a^b u(t)\,dt \cdot \int_a^b v(s)\,ds\right]}_{0},
\end{aligned}$$

so that

$$\left|\int_a^b w(t)\,dt\right|^2 = \int_a^b \int_a^b \left[u(t)u(s) + v(t)v(s)\right]\,dt\,ds. \qquad (*)$$

Now, if a, b, c, and d are real numbers, we have

$$0 \le (ad - bc)^2,$$

so that

$$2abcd \le a^2d^2 + b^2c^2,$$

and adding $a^2c^2 + b^2d^2$ to both sides of this inequality, we have

$$a^2c^2 + 2abcd + b^2d^2 \le a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2,$$

that is,

$$(ac + bd)^2 \le (a^2 + b^2)(c^2 + d^2),$$

which is just the Cauchy-Schwarz inequality in $\mathbb{R}^2$.

Applying this inequality to the integrand on the right hand side of $(*)$, we have

$$\begin{aligned} \left| \int_a^b w(t)\,dt \right|^2 &= \int_a^b \int_a^b \left[u(t)u(s) + v(t)v(s) \right]\,dt\,ds \\ &\le \int_a^b \int_a^b \sqrt{u(t)^2 + v(t)^2} \cdot \sqrt{u(s)^2 + v(s)^2}\,dt\,ds, \end{aligned}$$

that is,

$$\begin{aligned} \left| \int_a^b w(t)\,dt \right|^2 &\le \int_a^b \sqrt{u(t)^2 + v(t)^2}\,dt \cdot \int_a^b \sqrt{u(s)^2 + v(s)^2}\,ds \\ &= \left(\int_a^b \sqrt{u(t)^2 + v(t)^2}\,dt \right)^2 \\ &= \left(\int_a^b |w(t)|\,dt \right)^2. \end{aligned}$$

Finally, taking nonnegative square roots in this inequality, we have

$$\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt.$$

$\square$

• Note that we have used the fact that for a complex valued function of a real variable, we can write

$$\overline{\int_a^b w(s)\,ds} = \int_a^b \overline{w(s)}\,ds,$$

which is easily verified by considering the Riemann sums of the real and imaginary parts of the integrand.

• Note also that we wrote the double integral as an iterated integral,

$$\int_a^b \int_a^b w(t)\,\overline{w(s)}\,dt\,ds = \int_a^b w(t)\,dt \cdot \int_a^b \overline{w(s)}\,ds,$$

and this is easily justified, since the integrand on the left is separable.

Now we can give the result for complex valued functions of a complex variable.

Theorem. Let C be a contour of length L and $f(z)$ a piecewise continuous function on C, If M is a nonnegative constant such that

$$|f(z)| \leq M$$

for all points z on C at which $f(z)$ is defined, then

$$\left| \int_C f(z)\, dz \right| \leq M \cdot \int_a^b |z'(t)|\; dt = M \cdot L.$$

Proof. Let $z = z(t)$, $a \leq t \leq b$, be a parametric representation of C, from the previous theorem we have

$$\begin{aligned}
\left| \int_C f(z)\, dz \right| &= \left| \int_a^b f(z(t)) \cdot z'(t)\, dt \right| \\
&\leq \int_a^b |f(z(t)) \cdot z'(t)|\; dt \\
&= \int_a^b |f(z(t))| \cdot |z'(t)|\; dt \\
&\leq M \cdot \int_a^b |z'(t)|\; dt \\
&= M \cdot L,
\end{aligned}$$

and therefore,

$$\left| \int_C f(z)\, dz \right| \leq M \cdot \int_a^b |z'(t)|\; dt = M \cdot L.$$

$\square$