MATH 300 Spring - Summer 2018
Advanced Boundary Value Problems I
Solutions to Problem Set 4

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Question 1.

Solve the vibrating circular membrane problem for the radially symmetric case, that is, solve the initial value – boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{4}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad 0 \le r \le 1,\ t \ge 0$$

$$u(1,t) = 0,$$

$$u(r,0) = 5J_0(z_3 r)$$

$$\frac{\partial u}{\partial t}(r,0) = 0, \quad 0 \le r \le 1,$$

where $J_0(z)$ denotes the Bessel function of the first kind of order zero, and z_n denotes the n^{th} zero of $J_0(z)$.

Solution: We use separation of variables, assuming $u(r,t) = \phi(r)\cdot T(t)$, the wave equation above becomes

$$\frac{4}{r}(r\phi')'\cdot T = \phi\cdot T'',$$

and dividing by $4\,\phi\cdot T$, the variables are separated, and we get

$$\frac{(r\phi'(r))'}{r\phi(r)} = \frac{T''(t)}{4\,T(t)}.$$

The two sides of this equation must be a constant, say $-\lambda$, which yields two ordinary differential equations

$$(r\phi')' + \lambda\, r\,\phi = 0, \quad 0 \le r \le 1$$

$$T'' + 4\lambda\, T = 0, \quad t \ge 0.$$

The boundary condition $u(1,t) = 0$ for all $t \ge 0$ is satisfied if we require

$$\phi(1) = 0.$$

Also, since $r = 0$ is a singular point of the differential equation for ϕ, we add the requirement

$$|\phi(r)| \quad \text{bounded at} \quad r = 0,$$

which is equivalent to requiring that $|u(r,t)|$ be bounded at $r = 0$.

Thus, ϕ satisfies the boundary value problem

$$(r\phi')' + \lambda\, r\,\phi = 0, \quad 0 \le r \le 1$$

$$\phi(1) = 0,$$

$$|\phi(r)| \quad \text{bounded at} \quad r = 0.$$

We multiply the equation by r and recognize the equation

$$r^2\phi'' + r\phi' + \lambda r^2\phi = 0$$

as Bessel's equation of order zero, of which the function

$$\phi(r) = J_0(\sqrt{\lambda}\, r)$$

is the solution bounded at $r = 0$.

In order to satisfy the boundary condition $\phi(1) = 0$, we must have

$$J_0(\sqrt{\lambda}) = 0,$$

or

$$\sqrt{\lambda_n} = z_n, \quad n = 1,\, 2,\, \ldots$$

where z_n are the zeros of the function J_0.

Therefore the eigenfunctions and eigenvalues of the boundary value problem satisfied by $\phi(r)$ are

$$\phi_n(r) = J_0(\sqrt{\lambda_n}\, r) \qquad \text{and} \qquad \lambda_n = z_n^2$$

for $n \geq 1$.

For these values of λ which give a nontrivial solution to the boundary value problem for ϕ, the differential equation for T is

$$T''(t) + 4\lambda_n\, T(t) = 0$$

with general solution

$$T_n(t) = a_n \cos 2\sqrt{\lambda_n}\, t + b_n \sin 2\sqrt{\lambda_n}\, t,$$

for $n \geq 1$.

For each $n \geq 1$, the product solution

$$u_n(r,t) = \phi_n(r) \cdot T_n(t)$$

to the original partial differential equation satisfies the boundary condition $u(1,t) = 0$ and the boundedness condition $|u(r,t)|$ bounded at $r = 0$ for all $t \geq 0$.

Using the superposition principle we write the solution as

$$u(r,t) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_n}\, r) \left[a_n \cos 2\sqrt{\lambda_n}\, t + b_n \sin 2\sqrt{\lambda_n}\, t\right].$$

The initial conditions are satisfied if

$$u(r,0) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r) = 5J_0\left(z_3\, r\right)$$

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} 2b_n \sqrt{\lambda_n} J_0(\sqrt{\lambda_n}\, r) = 0,$$

for $0 \leq r \leq 1$.

Using the fact that the eigenfunctions $\{J_0(\sqrt{\lambda_n}\, r)\}_{n\geq 1}$ are orthogonal on the interval $[0,1]$ with respect to the weight function $\sigma(r) = r$, we see that $a_n = 0$ for all $n \neq 3$, and $a_3 = 5$, while $b_n = 0$ for all $n \geq 1$. Therefore the solution is

$$u(r,t) = 5J_0\left(z_3\, r\right) \cos 2z_3 t$$

for $0 \leq r \leq 1,\ t \geq 0$, where z_3 is the third zero of $J_0(z)$.

Question 2. [p 352, #8.2.2(b)]

Solve the heat equation with time-dependent sources and boundary conditions:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$u(x,0) = f(x)$$

Reduce the problem to one with homogeneous boundary conditions if

$$u(0,t) = A(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = B(t).$$

Hint: First use $u(x,t) = w(x,t) + v(x,t)$ and assume that v satisfies just the boundary conditions (and nothing else), then use the method of eigenfunction expansions to solve for $w(x,t)$.

Solution: If $u(x,t)$ is a solution to the problem $(*)$, we reduce the problem to one with homogeneous boundary conditions by writing

$$u(x,t) = v(x,t) + w(x,t)$$

where $v(x,t)$ satisfies only the boundary conditions

$$v(0,t) = A(t) \qquad (**)$$

$$\frac{\partial v}{\partial x}(L,t) = B(t)$$

for $t \geq 0$. We take the simplest possible such function, namely,

$$v(x,t) = B(t)\,x + A(t),$$

then

$$u(x,t) = w(x,t) + B(t)\,x + A(t),$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} + \frac{dB(t)}{dt}\,x + \frac{dA(t)}{dt},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}.$$

Therefore

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} - \frac{dB(t)}{dt}\,x - \frac{dA(t)}{dt} + Q(x,t).$$

Also,

$$A(t) = u(0,t) = w(0,t) + A(t) \qquad \text{so that} \qquad w(0,t) = 0,$$

while

$$B(t) = \frac{\partial u}{\partial x}(L,t) = \frac{\partial w}{\partial x}(L,t) + B(t) \qquad \text{so that} \qquad \frac{\partial w}{\partial x}(L,t) = 0.$$

Therefore, $w(x,t)$ satisfies the problem with homogeneous boundary condtions given by

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} - \frac{dB(t)}{dt}\,x - \frac{dA(t)}{dt} + Q(x,t), \quad 0 \leq x \leq L,\ t \geq 0 \qquad (***)$$

$$w(0,t) = 0, \quad t \geq 0$$

$$\frac{\partial w}{\partial x}(L,t) = 0, \quad t \geq 0$$

$$w(x,0) = f(x) - B(0)\,x - A(0), \quad 0 \leq x \leq L.$$

The initial value – boundary value problem for $w(x,t)$ is now a nonhomogeneous equation, however, it has homogeneous boundary conditions.

As is usual with nonhomogeneous equations, we first find the solution to the homogeneous problem

$$\begin{aligned} \frac{\partial w}{\partial t} &= k\frac{\partial^2 w}{\partial x^2} \\ w(0,t) &= 0 \\ \frac{\partial w}{\partial x}(L,t) &= 0 \end{aligned}$$

using separation of variables. Assuming a solution of the form $w(x,t) = \phi(x)\cdot T(t)$, we get two ordinary differential equations:

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0, \quad 0 \le x \le L, \qquad T'(t) + \lambda k\, T(t) = 0, \quad t \ge 0, \\ \phi(0) &= 0 \\ \phi'(L) &= 0 \end{aligned}$$

The eigenvalues are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$$

with corresponding eigenfunctions

$$\phi_n(x) = \sin\tfrac{(2n-1)\pi}{2L}x$$

for $n \ge 1$.

Now, we are not solving the T equation and finding the general solution to the homogeneous problem, instead we use the method of eigenfunction expansions to write the solution $w(x,t)$ to $(***)$, the nonhomogeneous problem, in terms of the eigenfunctions of the homogeneous problem:

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\tfrac{(2n-1)\pi}{2L}x, \tag{\dagger}$$

where (similar to the method of variation of parameters) the coefficients $a_n(t)$ depend on t.

Next, we force this to be a solution to the equation $(***)$ by requiring that each $a_n(t)$ satisfies a first-order ordinary differential equation together with an initial condition. We look at the initial conditions first, when $t = 0$ we want

$$w(x,0) = f(x) - B(0)\,x - A(0) = \sum_{n=1}^{\infty} a_n(0)\sin\tfrac{(2n-1)\pi}{2L}x,$$

and from the orthogonality of the eigenfunctions on the interval $[0,L]$, we need

$$a_n(0) = \frac{2}{L}\int_0^L [f(x) - B(0)\,x - A(0)]\sin\tfrac{(2n-1)\pi}{2L}x\,dx$$

for $n \ge 1$.

Now from (†) we have

$$\frac{\partial w}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n(t)}{dt} \sin \tfrac{(2n-1)\pi}{2L} x \qquad \text{and} \qquad \frac{\partial^2 w}{\partial x^2} = -\sum_{n=1}^{\infty} a_n(t) \left(\tfrac{(2n-1)\pi}{2L}\right)^2 \sin \tfrac{(2n-1)\pi}{2L} x,$$

and substituting these expressions into the equation $(***)$, after some simplification, we obtain

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} + k\lambda_n \, a_n \right] \sin \tfrac{(2n-1)\pi}{2L} x = -\frac{dB(t)}{dt} \, x - \frac{dA(t)}{dt} + Q(x,t).$$

The left-hand side of this equation is just the generalized Fourier series of the function

$$g(x,t) = -\frac{dB(t)}{dt} \, x - \frac{dA(t)}{dt} + Q(x,t),$$

so that

$$\frac{da_n}{dt} + k\lambda_n \, a_n = \frac{2}{L} \int_0^L g(x,t) \sin \tfrac{(2n-1)\pi}{2L} x \, dx = G_n(t), \tag{††}$$

and $a_n(t)$ satisfies the initial-value problem

$$\frac{da_n(t)}{dt} + k\lambda_n \, a_n(t) = G_n(t), \quad t \geq 0$$

$$a_n(0) = \frac{2}{L} \int_0^L [f(x) - B(0)\, x - A(0)] \sin \tfrac{(2n-1)\pi}{2L} x \, dx.$$

Multiplying by the integrating factor $e^{\lambda_n kt}$, this we can solve this first-order linear equation to get

$$a_n(t) = a_n(0) e^{-\lambda_n kt} + e^{-\lambda_n kt} \int_0^t G_n(s) e^{\lambda_n ks} \, ds, \quad t \geq 0$$

for $n \geq 1$.

The solution to the original equation is

$$u(x,t) = B(t)\, x + A(t) + \sum_{n=1}^{\infty} a_n(t) \sin \sqrt{\lambda_n}\, x$$

for $0 \leq x \leq L,\ t \geq 0$, where

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L} \right)^2$$

for $n \geq 1$.

Question 3. [p 353, #8.2.3]

Solve the two-dimensional heat equation with circularly symmetric time-independent sources, boundary conditions, and initial conditions (inside a circle):

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with

$$u(r,0) = f(r) \qquad \text{and} \qquad u(a,t) = T.$$

Solution: We first convert the problem into one with homogeneous boundary conditions and then use the method of egenfunction expansions to solve the nonhomogeneous equation that results.

Step 1: In order to get a problem with homogeneous boundary conditions we write

$$u(r,t) = v(r) + w(r,t)$$

where $v(r)$, the steady-state or equilibrium solution, satisfies

$$\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0, \quad 0 \le r \le a,$$

$$v(a) = T, \quad t \ge 0,$$

then

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} = \frac{\partial w}{\partial t},$$

and

$$\nabla^2 u = \nabla^2 v + \nabla^2 w = \nabla^2 w.$$

Therefore, $w(r,t)$ satisfies the initial value – boundary value problem

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r)$$

$$w(a,t) = 0$$

$$w(r,0) = f(r) - v(r)$$

We solve the v-problem first to get the constant solution

$$v(r) = T$$

for $0 \le r \le a$, so that $u(r,t) = w(r,t) + T$, and w satisfies the nonhomogeneous equation with homogeneous boundary conditions:

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r) \qquad (*)$$

$$w(a,t) = 0$$

$$w(r,0) = f(r) - T.$$

Step 2: Next we find the eigenvalues and eigenfunctions for the corresponding homogeneous problem:

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) \qquad (**)$$

$$w(a,t) = 0$$

$$|w(r,t)| \quad \text{bounded at} \quad r = 0,$$

were we have imposed the boundedness condition on physical grounds.

Using separation of variables, we assume that $w(r,t) = \phi(r) \cdot T(t)$, and separating variables we get

$$(r\phi')' + \lambda\, r\, \phi = 0, \quad 0 \le r \le a$$

$$T' + \lambda\, k\, T = 0, \quad t \ge 0.$$

The boundary condition $u(a,t) = 0$ for all $t \ge 0$ is satisfied if we require

$$\phi(a) = 0.$$

Also, since $r = 0$ is a singular point of the differential equation for ϕ, we add the requirement

$$|\phi(r)| \quad \text{bounded at} \quad r = 0,$$

which is equivalent to requiring that $|w(r,t)|$ be bounded at $r = 0$.

Thus, ϕ satisfies the boundary value problem

$$(r\phi')' + \lambda\, r\, \phi = 0, \quad 0 \le r \le a$$

$$\phi(a) = 0,$$

$$|\phi(r)| \quad \text{bounded at} \quad r = 0.$$

We multiply the equation by r and recognize the equation

$$r^2\phi'' + r\phi' + \lambda\, r^2\phi = 0$$

as Bessel's equation of order zero, of which the function

$$\phi(r) = J_0(\sqrt{\lambda}\, r)$$

is the solution bounded at $r = 0$.

In order to satisfy the boundary condition $\phi(a) = 0$, we must have

$$J_0(\sqrt{\lambda}\, a) = 0,$$

or

$$\sqrt{\lambda_n}\, a = z_n, \quad n = 1, 2, \ldots$$

where z_n are the zeros of the function J_0.

Therefore the eigenfunctions and eigenvalues of the boundary value problem satisfied by $\phi(r)$ are

$$\phi_n(r) = J_0(\sqrt{\lambda_n}\, r) \qquad \text{and} \qquad \lambda_n = \frac{z_n^2}{a^2}$$

for $n \ge 1$.

Step 3: Now we use an eigenfunction expansion for $w(r,t)$ as

$$w(r,t) = \sum_{n=1}^{\infty} a_n(t) J_0(\sqrt{\lambda_n}\, r)$$

and determine the coefficients $a_n(t)$ so that $w(r,t)$ is a solution to the nonhomogeneous equation

$$\frac{\partial w}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + Q(r) \qquad (*)$$

$$w(a,t) = 0$$

$$w(r,0) = f(r) - T,$$

and this means we will need the Fourier-Bessel Series for $Q(r)$ and $f(r) - T$:

$$Q(r) = \sum_{n=1}^{\infty} q_n J_0(\sqrt{\lambda_n}\, r), \qquad \text{with} \qquad q_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r) Q(r) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr}$$

$$f(r) - T = \sum_{n=1}^{\infty} f_n J_0(\sqrt{\lambda_n}\, r), \qquad \text{with} \qquad f_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)(f(r) - T) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr}.$$

Substituting these expansions into $(*)$, we have

$$\sum_{n=1}^{\infty} \frac{da_n(t)}{dt} J_0(\sqrt{\lambda_n}\, r) = \sum_{n=1}^{\infty} a_n(t)(-\lambda_n) J_0(\sqrt{\lambda_n}\, r) + \sum_{n=1}^{\infty} q_n J_0(\sqrt{\lambda_n}\, r),$$

and using the orthogonality of the eigenfunctions, the coefficients $a_n(t)$ satisfy the linear differential equation

$$\frac{da_n(t)}{dt} + \lambda_n a_n(t) = q_n, \quad t \geq 0$$

for $n \geq 1$.

From the initial condition

$$w(r,0) = \sum_{n=1}^{\infty} a_n(0) J_0(\sqrt{\lambda_n}\, r) = f(r) - T = \sum_{n=1}^{\infty} f_n J_0(\sqrt{\lambda_n}\, r),$$

using the orthogonality again, we have

$$a_n(0) = f_n$$

for $n \geq 1$.

Therefore, $a_n(t)$ satisfies the initial value problem

$$\frac{da_n(t)}{dt} + \lambda_n a_n(t) = q_n, \quad t \geq 0$$

$$a_n(0) = f_n$$

for $n \geq 1$.

Multiplying by the integrating factor $e^{\lambda_n t}$, the differential equation becomes

$$\frac{d}{dt}\left(a_n(t)e^{\lambda_n t}\right) = q_n e^{\lambda_n t},$$

and integrating,

$$a_n(t)e^{\lambda_n t} - a_n(0) = \int_0^t q_n e^{\lambda_n s}\, ds,$$

so that

$$a_n(t) = a_n(0)e^{-\lambda_n t} + \int_0^t q_n e^{-\lambda_n (t-s)}\, ds = a_n(0)e^{-\lambda_n t} + \frac{q_n}{\lambda_n}\left(1 - e^{-\lambda_n t}\right), \quad t \geq 0$$

for $n \geq 1$.

<u>Step 4:</u> Putting everything together, the solution is

$$u(r,t) = v(r) + w(r,t) = T + \sum_{n=1}^{\infty} a_n(t) J_0(\sqrt{\lambda_n}\, r),$$

that is,

$$u(r,t) = T + \sum_{n=1}^{\infty}\left[\frac{q_n}{\lambda_n} + \left(f_n - \frac{q_n}{\lambda_n}\right)e^{-\lambda_n t}\right] J_0(\sqrt{\lambda_n}\, r)$$

for $0 \leq r \leq a,\ t \geq 0$, where $\lambda_n = \dfrac{z_n^2}{a^2}$, and

$$q_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r) Q(r) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr},$$

$$f_n = \frac{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)(f(r) - T) r\, dr}{\displaystyle\int_0^a J_0(\sqrt{\lambda_n}\, r)^2 r\, dr}$$

for $n \geq 1$.

Question 4. [p 358, #8.3.2]

Consider the heat equation with a steady source

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$

subject to the initial and boundary conditions:

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad \text{and} \qquad u(x,0) = f(x).$$

Obtain the solution by the method of eigenfunction expansion. Show that the solution approaches a steady-state solution.

SOLUTION: Since the problem already has homogeneous boundary conditions, we consider the corresponding homogeneous problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,\ t \ge 0$$

$$u(0,t) = 0, \quad t \ge 0$$

$$u(L,t) = 0, \quad t \ge 0.$$

The eigenvalues and eigenfunctions for this problem are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_n(x) = \sin \tfrac{n\pi}{L}x$$

for $n \ge 1$.

We write the solution to the nonhomogeneous problem as an expansion in terms of these eigenfunctions:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \tfrac{n\pi}{L}x,$$

and determine the coefficients $a_n(t)$ which force this to be a solution to the nonhomogeneous problem.

We will need the eigenfunction expansions for $Q(x)$ and $f(x)$:

$$Q(x) = \sum_{n=1}^{\infty} q_n \sin \tfrac{n\pi}{L}x, \qquad \text{with} \qquad q_n = \frac{2}{\pi}\int_0^L Q(x)\sin \tfrac{n\pi}{L}x\,dx$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \tfrac{n\pi}{L}x, \qquad \text{with} \qquad f_n = \frac{2}{\pi}\int_0^L f(x)\sin \tfrac{n\pi}{L}\,dx.$$

Substituting these expansions into the nonhomogeneous equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x),$$

we obtain

$$\sum_{n=1}^{\infty} \frac{d\,a_n(t)}{dt}\sin \tfrac{n\pi}{L}x = -\sum_{n=1}^{\infty} k\frac{n^2\pi^2}{L^2}a_n(t)\sin \tfrac{n\pi}{L}x + \sum_{n=1}^{\infty} q_n \sin \tfrac{n\pi}{L}x,$$

and using the orthogonality of the eigenfunctions on the interval $[0,L]$, the coefficients $a_n(t)$ satisfy the initial value problem

$$\frac{d\,a_n(t)}{dt} + \frac{n^2\pi^2}{L^2}k\,a_n(t) = q_n, \quad t \ge 0$$

$$a_n(0) = f_n$$

for $n \ge 1$.

The solution to this initial value problem is

$$a_n(t) = f_n e^{-\frac{n^2\pi^2}{L^2}kt} + q_n \int_0^t e^{-\frac{n^2\pi^2}{L^2}k(t-s)}\,ds,$$

that is,

$$a_n(t) = \frac{q_n}{k\frac{n^2\pi^2}{L^2}} + \left(f_n - \frac{q_n}{k\frac{n^2\pi^2}{L^2}}\right) e^{-\frac{n^2\pi^2}{L^2}kt}, \quad t \geq 0$$

for $n \geq 1$.

Note that since $k > 0$, we have $\lim\limits_{t\to\infty} a_n(t) = \dfrac{q_n}{k\frac{n^2\pi^2}{L^2}}$ for $n \geq 1$.

The solution to the heat equation with a steady source is therefore

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{q_n}{k\frac{n^2\pi^2}{L^2}} + \left(f_n - \frac{q_n}{k\frac{n^2\pi^2}{L^2}}\right) e^{-\frac{n^2\pi^2}{L^2}kt} \right] \sin \tfrac{n\pi}{L} x$$

for $0 \leq x \leq L$ and $t \geq 0$.

For large value of t, this solution approaches $r(x)$ where

$$r(x) = \lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty} \frac{q_n}{k\frac{n^2\pi^2}{L^2}} \sin \tfrac{n\pi}{L} x$$

for $0 \leq x \leq L$, where

$$q_n = \frac{2}{\pi} \int_0^L Q(x) \sin \tfrac{n\pi}{L} x\, dx$$

for $n \geq 1$.

Differentiating this twice with respect to x, we see that

$$r''(x) = -\sum_{n=1}^{\infty} \frac{q_n}{k} \sin \tfrac{n\pi}{L} x = -\frac{1}{k}\, Q(x),$$

and since $r(0) = r(L) = 0$, then the function $r(x)$ satisfies the boundary value problem

$$k\frac{d^2r}{dx^2} + Q = 0, \quad 0 \leq x \leq L$$

$$r(0) = 0$$
$$r(L) = 0,$$

which is exactly the boundary value problem for the steady state solution, that is, $r(x)$ is the steady state or equilibrium solution to the original heat flow problem.

Question 5. [p 455, #10.3.1(a)]

(a) Show that the Fourier transform is a linear operator; that is, show that

$$\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] = c_1 F(\omega) + c_2 G(\omega)$$

(b) Show that $\mathcal{F}\left[f(x)g(x)\right] \neq F(\omega)G(\omega)$.

SOLUTION:

(a) If the Fourier transforms of f and g both exist, then

$$\begin{aligned}\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(c_1 f(x) + c_2 g(x)\right) e^{i\omega x}\, dx \\ &= \frac{c_1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx + \frac{c_2}{2\pi}\int_{-\infty}^{\infty} g(x) e^{i\omega x}\, dx \\ &= c_1 \mathcal{F}\left(f(x)\right) + c_2 \mathcal{F}\left(g(x)\right),\end{aligned}$$

that is,

$$\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] = c_1 \mathcal{F}\left(f(x)\right) + c_2 \mathcal{F}\left(g(x)\right)$$

and the Fourier transform is a linear operator.

(b) If f and g are functions such that $\mathcal{F}(f(x)) = F(\omega)$ and $\mathcal{F}(g(x)) = G(\omega)$ both exist, then

$$\mathcal{F}^{-1}\left(F(\omega)\, G(\omega)\right) = \frac{1}{2\pi} f * g = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) g(x-t)\, dt \neq f(x)\cdot g(x),$$

and taking the Fourier transform of both sides,

$$\mathcal{F}\left[f(x)g(x)\right] \neq F(\omega)G(\omega).$$

Question 6. [p 456, #10.3.5]

If $F(\omega)$ is the Fourier transform of $f(x)$, show that the inverse Fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$. This result is known as the **Shift Theorem** for Fourier transforms.

SOLUTION: We have

$$\begin{aligned}\mathcal{F}^{-1}\left(e^{i\omega\beta}F(\omega)\right) &= \int_{-\infty}^{\infty} F(\omega) e^{i\omega\beta} e^{-i\omega x}\, d\omega \\ &= \int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-\beta)}\, d\omega \\ &= f(x-\beta).\end{aligned}$$

Question 7. [p 469, #10.4.4]

(a) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = f(x), \quad -\infty < x < \infty.$$

(b) Does your solution suggest a simplifying transformation?

SOLUTION:

(a) If $u(x,t)$ is the solution to

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = f(x), \quad -\infty < x < \infty,$$

let

$$\widehat{u}(\omega,t) = \mathcal{F}(u(x,t)) \qquad \text{and} \qquad \widehat{u}(\omega,0) = \widehat{f}(\omega),$$

then $\widehat{u}(\omega,t)$ satisfies the initial value problem

$$\frac{\partial \widehat{u}}{\partial t} = -(k\omega^2 + \gamma)\widehat{u}, \quad t \geq 0$$
$$\widehat{u}(\omega,0) = \widehat{f}(\omega),$$

with solution

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)e^{-(k\omega^2+\gamma)t} = \widehat{f}(\omega)e^{-k\omega^2 t}e^{-\gamma t}.$$

The solution to the partial differential equation is

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left(\widehat{u}(\omega,t)\right) \\ &= \mathcal{F}^{-1}\left(\widehat{f}(\omega)e^{-k\omega^2 t}e^{-\gamma t}\right) \\ &= e^{-\gamma t}\mathcal{F}^{-1}\left(\widehat{f}(\omega)e^{-k\omega^2 t}\right) \qquad (\text{since} \quad \mathcal{F}^{-1} \quad \text{is linear}) \\ &= \frac{1}{2\pi}\, e^{-\gamma t}\, f * g\,(x,t) \end{aligned}$$

where

$$g(x,t) = \frac{e^{-x^2/4kt}}{\sqrt{4\pi kt}}.$$

Therefore

$$u(x,t) = e^{-\gamma t}\int_{-\infty}^{\infty} f(s)\frac{e^{-(x-s)^2/4kt}}{\sqrt{4\pi kt}}\, ds$$

for $-\infty < x < \infty$ and $t > 0$.

(b) If we multiply the solution above by $e^{\gamma t}$, we find

$$e^{\gamma t}u(x,t) = f * G\,(x,t),$$

which looks like the solution to a homogeneous heat equation.
Indeed, if we define

$$w(x,t) = e^{\gamma t}u(x,t),$$

then

$$\begin{aligned}\frac{\partial w}{\partial t} &= \gamma e^{\gamma t}u + e^{\gamma t}\frac{\partial u}{\partial t}\\ &= \gamma w + e^{\gamma t}\left(k\frac{\partial^2 u}{\partial x^2} - \gamma u\right)\\ &= \gamma w + k\frac{\partial^2 w}{\partial x^2} - \gamma w,\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial w}{\partial t} &= k\frac{\partial^2 w}{\partial x^2}\\ w(x,0) &= f(x)\end{aligned}$$

for $-\infty < x < \infty, \quad t > 0$.

Question 8. [p 480, #10.5.12]

Solve

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty,\ t > 0\\ \frac{\partial u}{\partial x}(0,t) &= 0, \quad t > 0\\ u(x,0) &= f(x), \quad 0 < x < \infty.\end{aligned}$$

Solution: Since the boundary condition is a Neumann condition, we use the Fourier cosine transform. Let

$$\widetilde{u}(\omega,t) = C(u(x,t)) = \frac{2}{\pi}\int_0^\infty u(x,t)\cos\omega x\,dx,$$

and

$$\widetilde{f}(\omega) = C(f(x)) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx,$$

then

$$C\left(\frac{\partial u}{\partial t}\right) = \frac{\partial \widetilde{u}}{\partial t}(\omega,t),$$

and

$$C\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{2}{\pi}\frac{\partial u}{\partial x}(0,t) - \omega^2\widetilde{u}(\omega,t),$$

and from the boundary condition, $\dfrac{\partial u}{\partial x}(0,t) = 0$, so that

$$C\left(\frac{\partial^2 u}{\partial x^2}\right) = -\omega^2\widetilde{u}(\omega,t).$$

After taking the Fourier cosine transform of both sides of the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

the transform $\widetilde{u}(\omega, t)$ satisfies the initial value problem

$$\frac{\partial \widetilde{u}}{\partial t}(\omega, t) + k\omega^2\widetilde{u}(\omega, t) = 0$$

$$\widetilde{u}(\omega, 0) = \widetilde{f}(\omega),$$

with solution

$$\widetilde{u}(\omega, t) = \widetilde{u}(\omega, 0)e^{-\omega^2 kt} = \widetilde{f}(\omega)e^{-\omega^2 kt}$$

for $-\infty < \omega < \infty$ and $t > 0$.

Therefore

$$u(x, t) = \int_0^\infty \widetilde{f}(\omega)e^{-\omega^2 kt} \cos\omega x \, d\omega$$

for $0 < x < \infty$ and $t > 0$.

Note that each of the functions $\widetilde{f}(\omega)$, $e^{-\omega^2 kt}$, and $\cos\omega x$ in the integrand is an odd function of ω, so that

$$\int_0^\infty \widetilde{f}(\omega)e^{-\omega^2 kt} \cos\omega x \, d\omega = \frac{1}{2}\int_{-\infty}^\infty \widetilde{f}(\omega)e^{-\omega^2 kt} \cos\omega x \, d\omega.$$

Since $\sin\omega x$ is an odd function of ω, then

$$\int_{-\infty}^\infty \widetilde{f}(\omega)e^{-\omega^2 kt} \sin\omega x \, d\omega = 0,$$

and we can write the solution $u(x, t)$ as

$$\begin{aligned} u(x, t) &= \frac{1}{2}\int_{-\infty}^\infty \widetilde{f}(\omega)e^{-\omega^2 kt}(\cos\omega x - i\sin\omega x)\, d\omega \\ &= \int_{-\infty}^\infty \frac{\widetilde{f}(\omega)}{2}e^{-\omega^2 kt}e^{-i\omega x}\, d\omega, \end{aligned}$$

that is,

$$u(x, t) = \mathcal{F}^{-1}\left(\frac{\widetilde{f}(\omega)}{2}e^{-\omega^2 kt}\right). \qquad (*)$$

Let f_{even} be the even extension of $f(x)$ to $(-\infty, \infty)$, then

$$\begin{aligned} \frac{\widetilde{f}(\omega)}{2} &= \frac{1}{2}\frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f_{\text{even}}(x)\cos\omega x\, dx \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f_{\text{even}}(x)(\cos\omega x + i\sin\omega x)\, dx \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f_{\text{even}}(x)e^{i\omega x}\, dx \\ &= \mathcal{F}\left(f_{\text{even}}(x)\right), \end{aligned}$$

so that

$$\frac{\widetilde{f}(\omega)}{2} = \mathcal{F}\left(f_{\text{even}}(x)\right). \qquad (**)$$

From $(*)$ and $(**)$ it follows that $u(x,t)$ is the solution to the initial value – boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0$$

$$u(x,0) = f_{\text{even}}(x), \quad 0 < x < \infty,$$

and therefore

$$u(x,t) = f_{\text{even}} * G\,(x,t)$$

where $G(x,t)$ is the heat kernel or Gaussian kernel

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-x^2/4kt}.$$

The solution is then

$$\begin{aligned} u(x,t) &= f_{\text{even}} * G\,(x,t) \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f_{\text{even}}(s) e^{-(x-s)^2/4kt}\, ds \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} f(s) \left(e^{-(x+s)^2/4kt} + e^{-(x-s)^2/4kt}\right) ds, \end{aligned}$$

so that

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} f(s) \left(e^{-(x+s)^2/4kt} + e^{-(x-s)^2/4kt}\right) ds$$

for $0 < x < \infty,\ t > 0$.