MATH 300 Fall 2007
Advanced Boundary Value Problems I
Solutions to Problem Set 3
To Be Completed by:
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Question 1.

Small vertical vibrations u of a uniform vibrating string which is initially at rest are governed by the initial value – boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad 0 < x < L$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0 \quad \text{for} \quad t > 0.$$

Show that

$$u(x,t) = \frac{1}{2}\Big[F(x-ct) + F(x+ct)\Big],$$

where $F(x)$ is the odd periodic extension of $f(x)$.

Hint: Use separation of variables to solve the problem and then use the addition formula for the sine function:

$$\sin a \cos b = \frac{1}{2}\big[\sin(a+b) + \sin(a-b)\big]$$

to write the solution in the form shown above.

Solution: We assume a solution of the form

$$u(x,t) = X(t) \cdot T(t)$$

and separating variables we have two ordinary differential equations

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < L \qquad T''(t) + \lambda c^2 T(t) = 0, \quad t > 0$$
$$X(0) = 0 \qquad T'(0) = 0.$$
$$X(L) = 0$$

The eigenvalues and eigenfunctions for the X-equation are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad X_n(x) = \sin \tfrac{n\pi}{L} x$$

for $n \geq 1$, and the corresponding solutions of the T-equation are

$$T_n(t) = a_n \cos \tfrac{n\pi}{L} ct + b_n \sin \tfrac{n\pi}{L} ct$$

for $n \geq 1$.

Using the superposition principle, we write the solution as

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos \tfrac{n\pi}{L} ct + b_n \sin \tfrac{n\pi}{L} ct) \sin \tfrac{n\pi}{L} x,$$

and

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} (-a_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi}{L} ct + b_n \tfrac{n\pi c}{L} \cos \tfrac{n\pi}{L} ct) \sin \tfrac{n\pi}{L} x.$$

We determine the coefficients using the initial conditions and the orthogonality of the eigenfunctions on the interval $[0, L]$.

From the first initial condition we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi}{L} x, \qquad (*)$$

so that

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi}{L} x \, dx$$

for $n \geq 1$.

From the second initial condition we have

$$0 = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi}{L} x,$$

so that $b_n = 0$ for $n \geq 1$.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi}{L} ct \sin \tfrac{n\pi}{L} x = \sum_{n=1}^{\infty} a_n \left\{ \frac{1}{2} \sin \tfrac{n\pi}{L} (x - ct) + \frac{1}{2} \sin \tfrac{n\pi}{L} (x + ct) \right\}$$

for $0 < x < L$ and $t > 0$.

Note that if $f \in PWS[0, L]$, that is, f is peicewise smooth on the interval $[0, L]$, then $(*)$ is the Fourier sine series for f, and converges for <u>all</u> real numbers x, and, except for at most countably many values of x, it converges to the odd periodic extension F of f.

Therefore, assuming the odd periodic extension of F is continuous, the solution is

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi}{L} (x - ct) + \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi}{L} (x + ct) = \frac{1}{2} F(x - ct) + \frac{1}{2} F(x + ct)$$

for $0 < x < L$ and $t > 0$.

Question 2.

Small vertical vibrations u of a uniform vibrating string which is initially unperturbed are governed by the initial value – boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad 0 < x < L$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad t > 0.$$

Show that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(s)\,ds,$$

where $G(x)$ is the odd periodic extension of $g(x)$.

Hint: Use separation of variables to solve the problem and then use the addition formula for the cosine function:

$$\sin a \sin b = \frac{1}{2}\big[\cos(a-b) - \cos(a+b)\big]$$

to write the solution in the form shown above.

Solution: We assume a solution of the form

$$u(x,t) = X(t) \cdot T(t)$$

and separating variables we have two ordinary differential equations

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < L \qquad T''(t) + \lambda c^2 T(t) = 0, \quad t > 0$$

$$X(0) = 0 \qquad T(0) = 0.$$

$$X(L) = 0$$

As before, the eigenvalues and eigenfunctions for the X-equation are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad X_n(x) = \sin \tfrac{n\pi}{L} x$$

for $n \geq 1$, and the corresponding solutions of the T-equation are

$$T_n(t) = a_n \cos \tfrac{n\pi}{L} ct + b_n \sin \tfrac{n\pi}{L} ct$$

for $n \geq 1$.

Using the superposition principle, we write the solution as

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos \tfrac{n\pi}{L} ct + b_n \sin \tfrac{n\pi}{L} ct) \sin \tfrac{n\pi}{L} x,$$

and

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} (-a_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi}{L} ct + b_n \tfrac{n\pi c}{L} \cos \tfrac{n\pi}{L} ct) \sin \tfrac{n\pi}{L} x.$$

We determine the coefficients using the initial conditions and the orthogonality of the eigenfunctions on the interval $[0, L]$.

From the first initial condition we have

$$0 = u(x,0) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi}{L} x,$$

so that $a_n = 0$ for $n \geq 1$.

From the second initial condition we have

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi}{L} x, \qquad (**)$$

so that

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \tfrac{n\pi}{L} x \, dx$$

for $n \geq 1$, and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi}{L} ct \sin \tfrac{n\pi}{L} x = \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{2} \cos \tfrac{n\pi}{L}(x-ct) - \frac{1}{2} \cos \tfrac{n\pi}{L}(x+ct) \right\}$$

for $0 < x < L$ and $t > 0$.

Now,

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi}{L}(x-ct) + \frac{1}{2} \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{L} \sin \tfrac{n\pi}{L}(x+ct)$$

and if $g \in PWS[0,L]$, that is, g is peicewise smooth on the interval $[0,L]$, then $(**)$ is the Fourier sine series for g, and converges for <u>all</u> real numbers x, and, except for at most countably many values of x, it converges to the odd periodic extension G of g, that is,

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2} G(x-ct) + \frac{1}{2} G(x+ct).$$

Integrating this from 0 to t, we have

$$\int_0^t \frac{\partial u}{\partial t}(x,\tau) \, d\tau = u(x,t) - u(x,0) = u(x,t)$$

since $u(x,0) = 0$. Therefore

$$u(x,t) = \frac{1}{2} \int_0^t G(x-c\tau) \, d\tau + \frac{1}{2} \int_0^t G(x+c\tau) \, d\tau = -\frac{1}{2c} \int_x^{x-ct} G(s) \, ds + \frac{1}{2c} \int_x^{x+ct} G(s) \, ds,$$

where we made the substitution $s = x - c\tau$ in the first integral, and $s = x + c\tau$ in the second integral, and assuming the odd periodic extension of g is continuous, the solution is

$$G(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(s) \, ds$$

for $0 < x < L$ and $t > 0$.

Question 3.

Using the one-dimensional wave equation governing the small vertical displacements of a uniform vibrating string,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0$$

derive the conservation of energy for a vibrating string,

$$\frac{dE}{dt} = \rho c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L,$$

where the total enery E is the sum of the kinetic energy and the potential energy,

$$E(t) = \frac{\rho}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{\rho c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

SOLUTION: The total energy (potential energy plus kinetic energy) of the string at time t is given by

$$E(t) = \frac{1}{2}\int_0^L \left[T\left(\frac{\partial u}{\partial x}\right)^2 + \rho\left(\frac{\partial u}{\partial t}\right)^2\right] dx = \frac{\rho}{2}\int_0^L \left[c^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] dx.$$

Using Leibniz's rule, we have

$$\begin{aligned}
E'(t) &= \frac{d}{dt}\left(\frac{\rho}{2}\int_0^L \left[c^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] dx\right) \\
&= \rho\int_0^L \left[c^2\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial t\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial t^2}\right] dx \\
&= \rho\int_0^L \left[c^2\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial t} + \frac{\partial u}{\partial t}\cdot c^2\frac{\partial^2 u}{\partial x^2}\right] dx \\
&= \rho c^2\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\right) dx \\
&= \rho c^2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\bigg|_0^L.
\end{aligned}$$

Note that if the string is fixed at both ends, so that

$$\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(L,t) = 0$$

for all $t > 0$, then $E'(t) = 0$ for all $t > 0$, that is, the total energy of the string is conserved.

Question 4.

Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + \big[\lambda\beta(x) + \gamma(x)\big]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + \big[\lambda\sigma(x) + q(x)\big]\phi = 0.$$

Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$?

Solution: Multiplying the differential equation by $H(x)$ we have

$$H\,\frac{d^2\phi}{dx^2} + \alpha\,H\,\frac{d\phi}{dx} + \lambda\,\beta\,H\,\phi + \gamma\,H\,\phi = 0,$$

and we want to determine H so that the first two terms are an exact derivative, that is,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] = H\,\frac{d^2\phi}{dx^2} + \alpha\,H\,\frac{d\phi}{dx},$$

that is,

$$p(x)\,\frac{d^2\phi}{dx^2} + \frac{dp(x)}{dx}\,\frac{d\phi}{dx} = H\,\frac{d^2\phi}{dx^2} + \alpha\,H\,\frac{d\phi}{dx}.$$

Thus, we want

$$p(x) = H(x) \qquad \text{and} \qquad p'(x) = \alpha(x)\,H$$

so that $H(x)$ satisfies the differential equation

$$H'(x) = \alpha(x)\,H(x).$$

If we take

$$p(x) = H(x) = e^{\int \alpha(x)\,dx},$$

then the differential equation is in Sturm-Liouville form

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + \big[\lambda\sigma(x) + q(x)\big]\phi = 0.$$

where

$$p(x) = e^{\int \alpha(x)\,dx}, \quad q(x) = \gamma(x)\,e^{\int \alpha(x)\,dx}, \quad \sigma(x) = \beta(x)\,e^{\int \alpha(x)\,dx}.$$

Note that $p(x) > 0$ and $\sigma(x) > 0$ provided that $\beta(x) > 0$.

Question 5.

Consider the partial differential equation which describes the temperature u in heat flow with convection:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}.$$

(a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm-Liouville form, and then put it in Sturm-Liouville form.

(b) Solve the initial value – boundary value problem

$$u(0,t) = 0, \quad t > 0$$
$$u(L,t) = 0, \quad t > 0$$
$$u(x,0) = f(x), \quad 0 < x < L$$

SOLUTION:

(a) Assuming a solution of the form $u(x,t) = \phi(x)\cdot h(t)$, the partial differential equation becomes

$$\phi h' = (k\phi'' - V_0\phi')h,$$

and separating variables

$$\frac{h'}{kh} = \frac{\phi''}{\phi} - \frac{V_0}{k}\frac{\phi'}{\phi} = -\lambda$$

where λ is the separation constant.

Thus, we have the following two ordinary differential equations

$$\phi''(x) - \frac{V_0}{k}\phi'(x) + \lambda\phi(x) = 0, \quad 0 < x < L$$

and

$$h'(t) + \lambda\, k\, h(t) = 0, \quad t > 0.$$

The spatial equation is not of the form

$$\frac{d}{dx}\Big(p(x)\phi'(x)\Big) + [q(x) + \lambda\sigma(x)]\phi(x) = 0, \quad 0 \le x \le L$$

where $p, q,$ and σ satisfy the conditions for a Sturm-Liouville problem, since we would need

$$p(x) = 1 \qquad \text{and} \qquad p'(x) = -\frac{V_0}{k},$$

and that doesn't work.

Multipying the spatial equation by $e^{-\frac{V_0 x}{k}}$, we have

$$\frac{d}{dx}\left(e^{-\frac{V_0 x}{k}}\frac{d\phi}{dx}\right) + \lambda e^{-\frac{V_0 x}{k}}\phi = 0,$$

which is in Sturm-Liouville form.

(b) The heat equation with convection satisfies the boundary value – initial value problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}, \quad 0 < x < L, \quad t > 0$$
$$u(0,t) = 0, \quad t > 0$$
$$u(L,t) = 0, \quad t > 0$$
$$u(x,0) = f(x), \ \ 0 < x < L$$

Assuming a solution of the form $u(x,t) = \phi(x) \cdot h(t)$ and separating variables, we get the two problems:

$$\phi''(x) - \frac{V_0}{k}\,\phi'(x) + \lambda\phi(x) = 0, \quad 0 < x < L \qquad\qquad h'(t) + \lambda\, k\, h(t) = 0, \quad t > 0,$$
$$\phi(0) = 0,$$
$$\phi(L) = 0,$$

Making the transformation

$$y = e^{-\frac{V_0 x}{2k}}\,\phi,$$

then y satisfies the boundary value problem

$$y'' + \left(\lambda - \frac{V_0^2}{4k^2}\right) y = 0, \quad 0 < x < L$$
$$y(0) = 0,$$
$$y(L) = 0,$$

which has nontrivial solutions if and only if

$$\lambda - \frac{V_0^2}{4k^2} > 0 \qquad \text{and} \qquad \lambda - \frac{V_0^2}{4k^2} = \frac{n^2\pi^2}{L^2}$$

for some integer $n \geq 1$, and the corresponding solutions are

$$y_n(x) = \sin\tfrac{n\pi}{L}x.$$

Therefore for the Sturm Liouville problem, the eigenvalues are

$$\lambda_n = \frac{V_0^2}{4k^2} + \frac{n^2\pi^2}{L^2}$$

with corresponding eigenfunctions

$$\phi_n(x) = e^{\frac{V_0 x}{2k}}\sin\tfrac{n\pi}{L}x, \quad 0 < x < L$$

for $n \geq 1$.

The corresponding solutions to the time equation are

$$h_n(t) = e^{-\lambda_n k t}$$

for $n \geq 1$.

For each $n \geq 1$, the products

$$u_n(x,t) = e^{\frac{V_0}{2k}x} e^{-\lambda_n kt} \sin \tfrac{n\pi x}{L}$$

satisfy the partial differential equation as well as the boundary conditions.

Using the superposition principle, we write

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{V_0}{2k}x} e^{-\lambda_n kt} \sin \tfrac{n\pi x}{L},$$

and we can satisfy the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n e^{\frac{V_0}{2k}x} \sin \tfrac{n\pi x}{L}$$

using the orthogonality of the eigenfunctions on the interval $[0, L]$ with respect to the weight function

$$\sigma(x) = e^{-\frac{V_0}{k}x}.$$

We have

$$b_n = \frac{2}{L} \int_0^L f(x) e^{-\frac{V_0}{2k}x} \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geq 1$.

Therefore the solution to the initial value – boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{V_0}{2k}x} e^{-\lambda_n kt} \sin \tfrac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) e^{-\frac{V_0}{2k}x} \sin \tfrac{n\pi x}{L}\, dx$$

for $n \geq 1$.

Question 6.

For the Sturm-Liouville eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad 0 < x < L$$

$$\frac{d\phi}{dx}(0) = 0$$

$$\phi(L) = 0,$$

verify the following general properties:

(a) There are an infinite number of eigenvalues with a smallest but no largest.

(b) The n^{th} eigenfunction has $n-1$ zeros.

(c) The eigenfunctions are complete and orthogonal.

(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

Solution:

(a) Assuming that the eigenvalues are real, we have to consider the three cases when $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

<u>case 1</u>: If $\lambda = 0$, the general solution to the differential equation $\phi''(x) = 0$ is $\phi(x) = Ax + B$, with $\phi'(x) = A$, and applying the first boundary condition $\phi'(0) = 0$, we have $A = 0$, and the solution is $\phi(x) = B$ for $0 < x < L$. Applying the second boundary condition $\phi(L) = 0$, we have $B = 0$, and the only solution in this case is the trivial solution $\phi(x) = 0$ for $0 < x < L$. Therefore $\lambda = 0$ is not an eigenvalue.

<u>case 2</u>: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' - \mu^2\phi = 0$ is

$$\phi(x) = A\cosh\mu x + B\sinh\mu x \qquad \text{with} \qquad \phi'(x) = \mu A\sinh\mu x + \mu B\cosh\mu x.$$

Applying the first boundary condition $\phi'(0) = \mu B = 0$ implies that $B = 0$, and the solution is now

$$\phi(x) = A\cosh\mu x$$

Applying the second boundary condition $\phi(L) = 0$ implies that $A\cosh\mu L = 0$, so that $A = 0$, and in this case we have only the trivial solution $\phi(x) = 0$ for $0 < x < L$.

<u>case 3</u>: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' + \mu^2\phi = 0$ is

$$\phi(x) = A\cos\mu x + B\sin\mu x \qquad \text{with} \qquad \phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x.$$

Applying the first boundary condition $\phi'(0) = 0$ implies that $\mu B = 0$, so that $B = 0$, and the solution is now

$$\phi(x) = A\cos\mu x$$

Applying the second boundary condition $\phi(L) = 0$ implies that $A\cos\mu L = 0$, and if $A = 0$ we get only the trivial solution. The boundary value problem has a nontrivial solution if and only if $\cos\mu L = 0$, that is, if and only if $\mu L = (n - \frac{1}{2})\pi$ for some integer $n \geq 1$, and therefore the eigenvalues are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$$

with corresponding eigenfunctions

$$\phi_n(x) = \cos\tfrac{(2n-1)\pi x}{2L}$$

for $n = 1, 2, \ldots$.

The eigenvalues are therefore ordered as

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

and there are an infinite number of eigenvalues with the smallest one being $\lambda_1 = \dfrac{\pi^2}{4L^2}$, but there is no largest eigenvalue.

(b) For $n \geq 1$, the eigenfunction ϕ_n is given by

$$\phi_n(x) = \cos\tfrac{(2n-1)\pi x}{2L}$$

for $0 < x < L$. Note that

$$\phi_n(0) = 1 \qquad \text{while} \qquad \phi_n(L) = \cos\tfrac{(2n-1)\pi}{2} = 0,$$

and all the zeros of ϕ_n occur in the interval $(0, L]$.

Also, $\phi_n(x) = 0$ exactly when

$$\frac{(2n-1)\pi x}{2L} = \frac{(2k-1)\pi}{2}$$

for $1 \le k \le n$, that is,

$$x = \left(\frac{2k-1}{2n-1}\right) L$$

for $1 \le k \le n$, and the eigenfunction $\phi_n(x) = \cos\frac{(2n-1)\pi x}{2L}$ has exactly n zeros in the interval $(0, L]$, that is, $\phi_m(x)$ has exactly $n-1$ zeros in the interval $(0, L)$.

(c) From Dirichlet's theorem we know that every f in the linear space of all piecewise smooth functions on $[0, L]$ has a Fourier series expansion in terms of the eigenfunctions, that is, the eigenfunctions form a complete set in the linear space $\mathcal{PWS}[0, L]$. The eigenfunctions form what is usually called a **Schauder Basis** for the linear space $\mathcal{PWS}[0, L]$. Recall that a *basis* for a linear space required that each element in the space could be written uniquely as a *finite* linear combination of the basis vetors.

Finally, we note that

$$\int_0^L \phi_m(x)\phi_n(x)\,dx = \int_0^L \cos\tfrac{(2m-1)\pi x}{2L}\cos\tfrac{(2n-1)\pi x}{2L}\,dx = 0$$

for $m, n \ge 1$ with $m \ne n$, and the set of eigenfunctions forms an orthogonal set.

(d) Using the boundary conditions

$$\phi'(0) = 0 \qquad \text{and} \qquad \phi(L) = 0$$

for the regular Sturm-Liouville problem above, we can write the eigenvalues in terms of the corresponding eigenfunctions as follows

$$\lambda_n = R(\phi_n) = \frac{\displaystyle\int_0^L \phi_n'(x)^2\,dx}{\displaystyle\int_0^L \phi_n(x)^2\,dx},$$

and clearly $\lambda_n \ge 0$.

If $\lambda_0 = 0$ is an eigenvalue then

$$\lambda_0 = R(\phi_0) = \frac{\displaystyle\int_0^L \phi_0'(x)^2\,dx}{\displaystyle\int_0^L \phi_0(x)^2\,dx} = 0,$$

and then $\phi_0'(x) = 0$ for $0 \le x \le L$, and $\phi_0(x)$ is a constant, and then $\phi_0(L) = 0$ implies that $\phi_0(x) = 0$ for $0 < x < L$, which is a contradiction, and therefore $\lambda_0 = 0$ is **not** an eigenvalue.

Question 7.

Show that $\lambda \ge 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi = 0, \quad 0 < x < 1$$

with

$$\frac{d\phi}{dx}(0) = 0 \qquad \text{and} \qquad \frac{d\phi}{dx}(1) = 0.$$

Solution: This is a regular Sturm-Liouville problem with

$$p(x) = 1, \qquad q(x) = -x^2 \le 0, \qquad \text{and} \qquad \sigma(x) = 1$$

for $0 \le x \le 1$, and from the boundary conditions

$$\Big[-p(x)\phi(x)\phi'(x) \Big] \Big|_0^1 = 0,$$

and the Rayleigh quotient reduces to

$$\lambda = R(\phi) = \frac{\displaystyle\int_0^1 \left[\phi'(x)^2 + x^2\phi(x)^2 \right] dx}{\displaystyle\int_0^1 \phi(x)^2\, dx} \ge 0,$$

and all of the eigenvalues are nonnegative.

If $\lambda = 0$ is an eigenvalue and ϕ_0 is the corresponding eigenfunction (and is thus not identically zero on the interval $[0, 1]$), then

$$0 = R(\phi_0) = \frac{\displaystyle\int_0^1 \left[\phi_0'(x)^2 + x^2\phi_0(x)^2 \right] dx}{\displaystyle\int_0^1 \phi_0(x)^2\, dx},$$

assuming that ϕ_0 and ϕ_0' are continuous on the interval $[0, 1]$, this implies that

$$\phi_0'(x)^2 = 0 \qquad \text{and} \qquad x^2\phi_0(x)^2 = 0$$

for all $x \in [0, 1]$, and this implies that $\phi_0(x) = 0$ for all $x \in [0, 1]$, which is a contradiction. Therefore $\lambda_0 = 0$ is **not** an eigenvalue.

Question 8.

Consider the initial value – boundary value problem

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x} \right) + \alpha u, \quad 0 < x < L, \;\; t > 0$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad t > 0$$

$$\frac{\partial u}{\partial x}(L, t) = 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad 0 < x < L,$$

where c, ρ, K_0, α are functions of x.

Assume that the appropriate eigenfunctions are known. Show that the eigenvalues are positive if $\alpha < 0$. Solve the initial value – boundary value problem, briefly discussing $\lim\limits_{t\to\infty} u(x, t)$.

Solution:

(a) We use separation of variables. Assume a solution of the form $u(x, t) = \phi(x)h(t)$ and substitute this into the partial differential equation to get

$$c\rho\phi h' = \left(K_0\phi' \right)' h + \alpha\phi h.$$

Separating variables,

$$\frac{(K_0\phi')'}{c\rho\phi} + \frac{\alpha}{c\rho} = \frac{h'}{h} = -\lambda$$

where λ is the separation constant.

This leads to the two ordinary differential equations:

$$\begin{aligned} (K_0(x)\phi'(x))' + \alpha(x)\phi(x) + \lambda c(x)\rho(x)\phi(x) &= 0, \quad 0 \le x \le L; \quad qh'(t) + \lambda h(t) = 0, \quad t > 0. \\ \phi'(0) &= 0, \\ \phi'(L) &= 0, \end{aligned}$$

The spatial equation is a regular Sturm-Liouville problem with $p(x) = K_0(x)$, $q(x) = \alpha(x)$, and $\sigma(x) = c(x)\rho(x)$, all of which are assumed continuous on the closed interval $[0, L]$. In addition, on physical grounds we assume that $K_0,\ c,$ and ρ are nonnegative and not identically zero on $[0, L]$.

If λ is an eigenvalue with corresponding eigenvector $\phi(x),\ 0 < x < L$, from the boundary conditions we have

$$\Big[- p(x)\phi(x)\phi'(x)\Big]\Big|_0^L = 0,$$

and the Rayleigh quotient reduces to

$$\lambda = R(\phi) = \frac{\displaystyle\int_0^L \left[K_0(x)\phi'(x)^2 - \alpha(x)\phi(x)^2\right]\, dx}{\displaystyle\int_0^L \phi(x)^2\rho(x)c(x)\, dx},$$

and if $\alpha(x) < 0$ for $0 \le x \le L$, then $\lambda \ge 0$.

Note that in this case, $\lambda = 0$ is impossible, since that would imply that $\phi(x) = 0$ for all $0 < x < L$, which is a contradiction. Therefore all the eigenvalues are strictly positive.

(b) The boundary value problem for ϕ is a regular Sturm-Liouville problem and has an infinite sequence of eigenvalues and corresponding eigenfunctions $\{(\lambda_n, \phi_n)\}_{n\ge1}$ where the ϕ_n's form a complete orthogonal set of functions in the linear space of piecewise continuous functions on $[0, L]$ with respect to the weight function $\sigma(x) = c(x)\rho(x)$.

The corresponding solutions to the time equation are

$$h_n(t) = c_n e^{-\lambda_n t}, \quad t > 0$$

for $n \ge 1$.

Using the superposition principle, we can write

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\phi_n(x)$$

for $0 < x < L,\ t > 0$, and this satifies the partial differential equation and the boundary conditions.

In order to satisfy the initial condition, we use the orthogonality of the eigenfunctions to write

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n\phi_n(x),$$

where

$$c_n = \frac{\displaystyle\int_0^L f(x)\phi_n(x)c(x)\rho(x)\, dx}{\displaystyle\int_0^L \phi_n(x)^2 c(x)\rho(x)\, dx}.$$

(c) Since $\lambda_n > 0$ for all $n \geq 0$, then for each term in the series,

$$e^{-\lambda_n t} \longrightarrow 0,$$

as $t \to \infty$, and therefore

$$\lim_{t\to\infty} u(x,t) = 0$$

for eaxh $x \in (0, L)$.

Question 9.

Give an example of an eigenvalue problem with more than one eigenfunction corresponding to an eigenvalue.

SOLUTION: Consider the boundary value problem with periodicity conditions as given below.

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad -\pi < x < \pi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{dx}(-\pi) = \frac{d\phi}{dx}(\pi).$$

The eigenvalues are $\lambda_n = n^2$ with corresponding eigenfunctions

$$\phi_n(x) = \cos nx \qquad \text{and} \qquad \psi_n(x) = \sin nx$$

for $n \geq 0$.

Therefore there are two linearly independent eigenfunctions for each eigenvalue λ_n for $n \geq 1$. For $\lambda_0 = 0$, there is only one eigenfunction, namely, $\phi_0(x) = 1$ for $-\pi < x < \pi$.

Question 10

Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $u\,L(v) - v\,L(u)$ is an exact differential.

(b) Evaluate

$$\int_0^1 [u\,L(v) - v\,L(u)]\,dx$$

in terms of the boundary data for any functions u and v.

(c) Show that

$$\int_0^1 [u\,L(v) - v\,L(u)]\,dx = 0$$

if u and v are any two functions satisfying the boundary conditions

$$\phi(0) = 0 \qquad \phi(1) = 0$$

$$\frac{d\phi}{dx}(0) = 0 \qquad \frac{d^2\phi}{dx^2}(1) = 0.$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [u\,L(v) - v\,L(u)]\;dx = 0.$$

(e) For the eigenvalue problem [using the boundary conditions in part (c)]

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0, \quad 0 < x < 1,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

SOLUTION:

(a) We consider

$$uv^{(4)} = (uv''')' - u'v''' = (uv''')' - (u'v'')' + u''v'',$$

that is,

$$uv^{(4)} = (uv''')' - (u'v'')' + u''v''. \tag{$*$}$$

By symmetry,

$$vu^{(4)} = (vu''')' - (v'u'')' + v''u'', \tag{$**$}$$

and subtracting $(**)$ from $(*)$ we have

$$uL(v) - vL(u) = \left(uv''' - vu''' - u'v'' + v'u''\right)',$$

and $uL(v) - vL(u)$ is an exact differential.

(b) We have

$$\begin{aligned}\int_0^1 [uL(v) - vL(u)]\;dx = \Big[uv''' - vu''' - u'v'' + v'u''\Big]\Big|_0^1 \\ = \quad & u(1)v'''(1) - v(1)u'''(1) - u'(1)v''(1) + v'(1)u''(1) \\ & - u(0)v'''(0) + v(0)u'''(0) + u'(0)v''(0) - v'(0)u''(0).\end{aligned}$$

(c) If u and v are any two functions satisfying the boundary conditions

$$\begin{aligned}\phi(0) = 0, \qquad & \phi(1) = 0, \\ \phi'(0) = 0, \qquad & \phi''(1) = 0.\end{aligned}$$

From part (b) each of the first four terms contains either $u(1)$, $v(1)$, $u''(1)$, or $v''(1)$, each of which is 0, while each of the last four terms contains either $u(0)$, $v(0)$, $u'(0)$, or $v'(0)$, each of which is also 0.

(d) Another set of boundary conditions for which

$$\int_0^L [uL(v) - vL(u)]\;dx = 0$$

is given by

$$\begin{aligned}\phi'(0) = 0, \qquad & \phi'(1) = 0, \\ \phi'''(0) = 0, \qquad & \phi'''(1) = 0.\end{aligned}$$

(e) Let (λ_n, ϕ_n) and (λ_m, ϕ_m) be distinct eigenvalue – eigenfunction pairs satisfying the boundary value problem

$$\begin{aligned} \frac{d^4\phi}{dx^4} + \lambda e^x \phi &= 0, \quad 0 < x < 1, \\ \phi(0) = 0, \qquad & \phi(1) = 0, \\ \phi'(0) = 0, \qquad & \phi''(1) = 0, \end{aligned}$$

then we have

$$\begin{aligned} 0 &= \int_0^1 \phi_n L(\phi_m) - \phi_m L(\phi_n)\, dx \\ &= \int_0^1 \left[\phi_n \left(-\lambda_m e^x \phi_m\right) - \phi_m \left(-\lambda_n e^x \phi_n\right)\right] dx \\ &= (\lambda_n - \lambda_m) \int_0^1 \phi_n \phi_m e^x \, dx \end{aligned}$$

and if $\lambda_n \neq \lambda_m$, then

$$\int_0^1 \phi_n \phi_m e^x \, dx = 0$$

and ϕ_n and ϕ_m are orthogonal on the interval $[0, 1]$ with respect to the weight function

$$\sigma(x) = e^x$$

for $x \in [0, 1]$.