Solutions to Problem Set1
Math 300 - Spring-Summer 2018
Completion Date: May 21

These problems are a review of the techniques used last year in Math 201.

Question 1.

Find the general solution to the differential equation

$$(e^x\varphi')' + \lambda^2 e^x \varphi = 0.$$

SOLUTION: We have

$$(e^x\varphi')' + \lambda^2 e^x \varphi = 0$$

if and only if

$$e^x\varphi'' + e^x\varphi' + \lambda^2 e^x \varphi = 0,$$

and since $e^x \neq 0$ for any real number x, then φ is a solution to the original equation if and only if

$$\varphi'' + \varphi' + \lambda^2 \varphi = 0,$$

and this constant coefficient equation has characteristic equation $m^2 + m + \lambda^2 = 0$ with roots

$$m_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1-4\lambda^2}, \qquad \text{and} \qquad m_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{1-4\lambda^2}.$$

Now, considering all possible cases:

Case 1: if $1 - 4\lambda^2 > 0$, that is, $-\frac{1}{2} < \lambda < \frac{1}{2}$ the general solution is

$$\varphi(x) = e^{-x/2}\left[c_1 \cosh\left(\frac{1}{2}\sqrt{1-4\lambda^2}\,x\right) + c_2 \sinh\left(\frac{1}{2}\sqrt{1-4\lambda^2}\,x\right)\right],$$

Case 2: if $1 - 4\lambda^2 = 0$, that is, $\lambda = \pm\frac{1}{2}$, the general solution is

$$\varphi(x) = e^{-x/2}[c_1 + c_2 x],$$

Case 3: if $1 - 4\lambda^2 < 0$, that is, $\lambda > \frac{1}{2}$ or $\lambda < -\frac{1}{2}$, the general solution is

$$\varphi(x) = e^{-x/2}\left[c_1 \cos\left(\frac{1}{2}\sqrt{4\lambda^2-1}\,x\right) + c_2 \sin\left(\frac{1}{2}\sqrt{4\lambda^2-1}\,x\right)\right],$$

where c_1 and c_2 are arbitrary constants.

Question 2.

Compare and contrast the form of the solutions of these three differential equations and their behavior as $t \to \infty$.

(a) $\dfrac{d^2u}{dt^2} + u = 0$ (b) $\dfrac{d^2u}{dt^2} = 0$ (c) $\dfrac{d^2u}{dt^2} - u = 0.$

SOLUTION:

(a) The general solution to the equation $\dfrac{d^2u}{dt^2} + u = 0$ is $u(t) = c_1 \cos t + c_2 \sin t$, and $\lim_{t\to\infty} u(t) = 0$ if $c_1 = c_2 = 0$ and does not exist otherwise.

(b) The general solution to the equation $\dfrac{d^2u}{dt^2} = 0$ is $u(t) = c_1 t + c_2$, and

$$\lim_{t\to\infty} u(t) = \begin{cases} +\infty & \text{if } \; c_1 > 0 \\ c_2 & \text{if } \; c_1 = 0 \\ -\infty & \text{if } \; c_1 < 0. \end{cases}$$

(c) The general solution to the equation $\dfrac{d^2u}{dt^2} - u = 0$ is $u(t) = c_1 e^t + c_2 e^{-t}$, and

$$\lim_{t\to\infty} u(t) = \begin{cases} +\infty & \text{if } \; c_1 > 0 \\ 0 & \text{if } \; c_1 = 0 \\ -\infty & \text{if } \; c_1 < 0. \end{cases}$$

Question 3.

Find the general solution to the following differential equation (λ is a constant). Use an "exponential" trial solution.

$$\frac{d^4u}{dx^4} - \lambda^4 u = 0.$$

SOLUTION: We assume a solution of the form $u(x) = e^{mx}$, then $m^4 - \lambda^4 = 0$ so that $m = \pm\lambda, \ \pm i\lambda$, and the general solution is

$$u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3 \cos \lambda x + c_4 \sin \lambda x.$$

Question 4.

One solution of the differential equation

$$\frac{d}{dx}\left(x\,\frac{du}{dx}\right) + \frac{4x^2 - 1}{4x}\,u = 0$$

is given by

$$u_1(x) = \frac{\cos x}{\sqrt{x}}.$$

Find a second independent solution.

SOLUTION: The differential equation can be written in the form

$$x\,\frac{d^2u}{dx^2}+\frac{du}{dx}+\frac{4x^2-1}{4x}\,u=0$$

and in standard form

$$\frac{d^2u}{dx^2}+\frac{1}{x}\,\frac{du}{dx}+\frac{4x^2-1}{4x^2}\,u=0.$$

If $u_2(x)=v(x)\,u_1(x)$, then the equation satisfied by $v(x)$ is

$$u_1v''+2v'u_1'+\frac{1}{x}\,v'u_1=0$$

and since

$$u_1=\frac{\cos x}{\sqrt{x}},$$

$$u_1'=-\frac{\sin x}{\sqrt{x}}-\frac{\cos x}{2x^{3/2}},$$

then

$$\frac{\cos x}{\sqrt{x}}\,v''-2\frac{\sin x}{\sqrt{x}}\,v'=0,$$

that is,

$$\frac{v''}{v'}=\frac{2\sin x}{\cos x},$$

so that

$$\log|v'|=-2\log|\cos x|,$$

that is,

$$v'=\sec^2 x \qquad \text{and} \qquad v(x)=\tan x.$$

Therefore

$$u_2(x)=v(x)u_1(x)=\tan x\,\frac{\cos x}{\sqrt{x}}=\frac{\sin x}{\sqrt{x}}.$$

Question 5.

Given the differential equation

$$\frac{d}{d\rho}\left(\rho\,\frac{d\varphi}{d\rho}\right)+\frac{4\lambda^2\rho^2-1}{4\rho}\,\varphi=0,$$

use the change of variable

$$\varphi(\rho)=\frac{v(\rho)}{\sqrt{\rho}}$$

to solve the equation.

SOLUTION: Differentiating $\varphi(\rho)$, we have

$$\frac{d\varphi}{d\rho}=\frac{v'(\rho)}{\sqrt{\rho}}-\frac{v(p)}{2\rho^{3/2}}$$

$$\rho\,\frac{d\varphi}{d\rho}=\sqrt{\rho}\,v'(\rho)-\frac{v(\rho)}{2\sqrt{\rho}}$$

$$\frac{d}{d\rho}\left(\rho\,\frac{\varphi}{d\rho}\right)=\sqrt{\rho}\,v''(\rho)+\frac{1}{4}\,\frac{v(\rho)}{\rho^{3/2}}.$$

From the differential equation, we have

$$\sqrt{\rho}v'' + \frac{v}{4\rho^{3/2}} + \frac{4\lambda^2\rho^2 - 1}{4\rho}\frac{v}{\sqrt{\rho}} = 0$$

which implies that

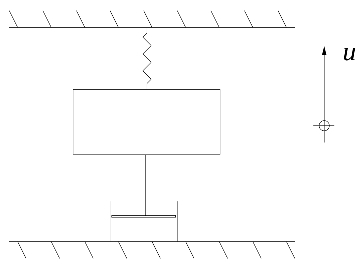
$$v'' + \lambda^2 v = 0,$$

with the general solution $v(x) = c_1 \cos\lambda\rho + c_2 \sin\lambda\rho$ and therefore

$$\varphi(\rho) = c_1 \frac{\cos\lambda\rho}{\sqrt{\rho}} + c_2 \frac{\sin\lambda\rho}{\sqrt{\rho}}.$$

Question 6.

The displacement $u(t)$ of a mass in mass-spring-damper system, as in the figure below,



is described by the initial value problem

$$\frac{d^2u}{dt^2} + b\frac{du}{dt} + \omega^2 u = 0$$
$$u(0) = u_0$$
$$\frac{du}{dt}(0) = v_0.$$

(The coefficients b and ω^2 are proportional to the characteristic constants of the damper and the spring, respectively.)

Solve the initial value problem for each of the parameter ranges below, and explain why these ranges might have been chosen.

(a) $b = 0$, (b) $0 < b < 2\omega$, (c) $b = 2\omega$, (d) $b > 2\omega$.

Solution: Assuming a solution to the differential equation

$$\frac{d^2u}{dt^2} + b\,\frac{du}{dt} + \omega^2 u = 0$$

of the form $u(t) = e^{mt}$, the characteristic equation is $m^2 + bm + \omega^2 = 0$ with roots

$$m = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4\omega^2}.$$

(a) If $b = 0$, then $m_1 = i\omega$ and $m_2 = -i\omega$, and the general solution is

$$u(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

Applying the initial conditions

$$u(0) = u_0 \quad \text{implies} \quad c_1 = u_0 \qquad \text{and} \qquad u'(0) = v_0 \quad \text{implies} \quad c_2 = \frac{v_0}{\omega},$$

so that the solution to the initial value problem is

$$u(t) = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t.$$

(b) If $0 < b < 2\omega$, then $m_1 = -\frac{b}{2} + \frac{i}{2}\sqrt{4\omega^2 - b^2}$, and $m_2 = -\frac{b}{2} - \frac{i}{2}\sqrt{4\omega^2 - b^2}$, and the general solution is

$$u(t) = e^{-\frac{b}{2}t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

where $\beta = \sqrt{\omega^2 - b^2/4}$. Applying the initial conditions

$$u(0) = u_0 \quad \text{implies} \quad c_1 = u_0 \qquad \text{and} \qquad u'(0) = v_0 \quad \text{implies} \quad c_2 = \frac{2v_0 + bu_0}{2\beta}$$

so that the solution to the initial value problem is

$$u(t) = e^{-\frac{b}{2}t}\left(u_0 \cos \beta t + \frac{2v_0 + bu_0}{2\beta} \sin \beta t\right).$$

(c) If $b = 2\omega$, then $m_1 = m_2 = -\frac{b}{2}$, and the general solution is

$$u(t) = (c_1 + c_2 t)e^{-\frac{b}{2}t}.$$

Applying the initial conditions

$$u(0) = u_0 \quad \text{implies} \quad c_1 = u_0 \qquad \text{and} \qquad u'(0) = v_0 \quad \text{implies} \quad c_2 = v_0 + u_0\omega$$

so that the solution to the initial value problem is

$$u(t) = (u_0 + (v_0 + u_0\omega)t)\, e^{-\frac{b}{2}t}.$$

(d) If $b > 2\omega$, then $m_1 = -\frac{b}{2} + \beta$, and $m_2 = -\frac{b}{2} - \beta$, where $\beta = \sqrt{b^2/4 - \omega}$, and the general solution is

$$u(t) = \left(c_1 e^{\beta t} + c_2 e^{-\beta t}\right)e^{-\frac{b}{2}t}.$$

Applying the initial conditions

$$u(0) = u_0 \quad \text{implies} \quad c_1 = \frac{1}{2\beta}(v_0 - m_2 u_0) \qquad \text{and} \qquad u'(0) = v_0 \quad \text{implies} \quad c_2 = \frac{1}{2\beta}(m_1 u_0 - v_0)$$

so that the solution to the initial value problem is

$$u(t) = \frac{1}{2\beta}\left[(v_0 - m_2 u_0)e^{m_1 t} + (m_1 u_0 - v_0)e^{m_2 t}\right].$$

It is clear from the above that these ranges of b were chosen because of the drastic change in behavior of the solutions in going from one interval to another.

Question 7.

Find the general solution to the differential equation

$$\frac{d^2u}{dx^2} - \gamma^2(u - U) = 0$$

where U and γ^2 are constant.

Solution: If we make the substitution $v = u - U$, then

$$\frac{d^2v}{dx^2} - \gamma^2 v = 0,$$

and the general solution to this equation is The general solution is

$$v(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}.$$

Therefore, the general solution to the original differential equation is

$$u(x) = U + c_1 e^{\gamma x} + c_2 e^{-\gamma x}.$$

Question 8.

Find the general solution of the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{du}{dr}\right) = -1.$$

Solution: Multiplying the differential equation by r, we have

$$\frac{d}{dr}\left(r\,\frac{du}{dr}\right) = -r,$$

and integrating,

$$r\,\frac{du}{dr} = -\frac{r^2}{2} + c_1,$$

so that

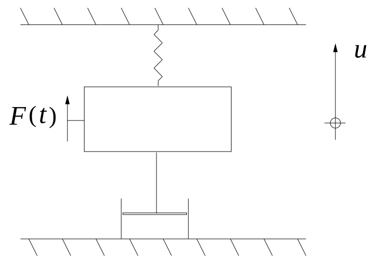
$$\frac{du}{dr} = -\frac{r}{2} + \frac{c_1}{r}.$$

Integrating again, we have

$$u(r) = -\frac{r^2}{4} + c_1 \log r + c_2.$$

Question 9.

The displacement $u(t)$ of a mass in mass-spring-damper system with an external force, as in the figure below,



is described by the initial value problem

$$\frac{d^2u}{dt^2} + b\frac{du}{dt} + \omega^2 u = f_0 \cos \mu t$$
$$u(0) = 0$$
$$\frac{du}{dt}(0) = 0.$$

(The coefficients b and ω^2 are proportional to the characteristic constants of the damper and the spring, respectively, and the coefficient f_0 is proportional to the amplitude of the external force.)

Solve the initial value problem for these three cases;

(a) $b = 0, \quad \mu \neq \omega,$ (b) $b = 0, \quad \mu = \omega,$ (c) $b > 0.$

SOLUTION:

(a) If $b = 0$, and $\mu \neq \omega$, the general solution to the homogeneous equation is given by

$$u_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

where c_1 and c_2 are arbitrary constants, and a particular solution to the nonhomogeneous equation is given by

$$u_p(t) = \frac{f_0}{\omega^2 - \mu^2} \cos \mu t.$$

The general solution to the nonhomogeneous equation is

$$u(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{f_0}{\omega^2 - \mu^2} \cos \mu t,$$

and applying the initial conditions

$$u(0) = 0 \quad \text{implies} \quad c_1 = -\frac{f_0}{\omega^2 - \mu^2} \qquad \text{and} \qquad u'(0) = 0 \quad \text{implies} \quad c_2 = 0,$$

and the solution to the initial value problem is

$$u(t) = \frac{f_0}{\omega^2 - \mu^2}(\cos \mu t - \cos \omega t).$$

(b) If $b = 0$ and $\mu = \omega$, the general solution to the homogeneous equation is given by

$$u_c(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

where c_1 and c_2 are arbitrary constants, and a particular solution to the nonhomogeneous equation is given by

$$u_p(t) = \frac{f_0 t}{2\omega} \sin \omega t.$$

The general solution to the nonhomogeneous equation is

$$u(t) = c_1 \cos \omega t + \left(c_2 + \frac{f_0 t}{2\omega}\right) \sin \omega t,$$

and applying the initial conditions

$$u(0) = 0 \quad \text{implies} \quad c_1 = 0 \qquad \text{and} \qquad u'(0) = 0 \quad \text{implies} \quad c_2 = 0,$$

and the solution to the initial value problem is

$$u(t) = \frac{f_0 t}{2\omega} \sin \omega t.$$

(c) If $b > 0$, the general solution to the homogeneous equation is

$$u_c(t) = (c_1 \cos \beta t + c_2 \sin \beta t)e^{-\frac{b}{2}t}, \quad \text{where} \quad \beta = \sqrt{\omega^2 - b^2/4}, \quad \text{if} \quad 0 < b < 2\omega,$$

$$u_c(t) = (c_1 + c_2 t)e^{-\frac{b}{2}t}, \quad \text{if} \quad b = 2\omega,$$

$$u_c(t) = (c_1 e^{\beta t} + c2e^{-\beta t})e^{-\frac{b}{2}t}, \quad \text{where} \quad \beta = \sqrt{b^2/4 - \omega^2}, \quad \text{if} \quad b > 2\omega$$

where c_1 and c_2 are arbitrary constants, and a particular solution to the nonhomogeneous equation is

$$u_p(t) = \frac{f_0}{(\omega^2 - \mu^2)^2 + \mu^2 b^2} \left[(\omega^2 - \mu^2) \cos \mu t + \mu b \sin \mu t\right].$$

Now,

$$u_c'(t) = \begin{cases} \left[\left(\beta c_2 - \dfrac{c_1 b}{2}\right) \cos \beta t - \left(\beta c_1 + \dfrac{c_2 b}{2}\right) \sin \beta t\right] e^{-\frac{b}{2}t} \\ \left[c_2 - \dfrac{b}{2}(c_1 + c_2 t)\right] e^{-\frac{b}{2}t} \\ \left[\left(\beta - \dfrac{b}{2}\right) c_1 e^{\beta t} - \left(\beta + \dfrac{b}{2}\right) c_2 e^{-\beta t}\right] e^{-\frac{b}{2}t} \end{cases}$$

in each of the respective cases above. Therefore,

$$u_c(0) = \begin{cases} c_1 \\ c_1 \\ c_1 + c_2 \end{cases}$$

and

$$u_c'(0) = \begin{cases} \beta c_2 - \dfrac{c_1 b}{2} \\ c_2 - \dfrac{b c_1}{2} \\ \left(\beta - \dfrac{b}{2}\right) c_1 - \left(\beta + \dfrac{b}{2}\right) c_2 \end{cases}$$

and

$$u_p(0) = \frac{f_0(\omega^2 - \mu^2)}{(\omega^2 - \mu^2)^2 + \mu^2 b^2} \qquad \text{and} \qquad u_p'(0) = \frac{f_0 \mu^2 b}{(\omega^2 - \mu^2)^2 + \mu^2 b^2}.$$

Since $u(t) = u_c(t) + u_p(t)$, applying the initial conditions, we have

$$u(0) = 0 \quad \text{implies} \quad u_c(0) = -u_p(0) \qquad \text{and} \qquad u'(0) = 0 \quad \text{implies} \quad u_c'(0) = -u_p'(0),$$

and

$$u(t) = u_c(t) + u_p(t)$$

with c_1 and c_2 as determined above.

Question 10.

Use variation of parameters to find a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \sin x,$$

if two independent solutions to the homogeneous equation are given by $y_1(x) = \cos x$, and $y_2(x) = \sin x$. Be sure that the differential equation is in the correct form.

Solution: The Wronskian is

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \cos^2 x + \sin^2 x = 1.$$

Writing

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

and forcing $y(x)$ to be a solution to the original differential equations gives

$$v_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\,dx = -\int \sin^2 x\,dx = -\frac{1}{2}\int (1 - \cos 2x)\,dx$$

and

$$v_1(x) = -\frac{1}{2}x + \frac{1}{4}\sin 2x.$$

Also

$$v_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\,dx = \int \cos x \sin x\,dx = \frac{1}{2}\sin^2 x,$$

and

$$v_2(x) = \frac{1}{2}\sin^2 x.$$

Therefore

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) = -\frac{1}{2}x\cos x + \frac{1}{4}\sin 2x \cos x + \frac{1}{2}\sin^3 x,$$

that is,

$$y_p(x) = -\frac{1}{2}x\cos x + \frac{1}{2}(\sin x \cos^2 x + \sin^3 x) = -\frac{1}{2}x\cos x + \frac{1}{2}\sin x$$

and we may take

$$y_p(x) = -\frac{1}{2}x\cos x.$$

Question 11.

Use variation of parameters to show that a particular solution of the differential equation

$$\frac{d^2u}{dt^2} - \gamma^2 u = f(t)$$

is given by

$$u_p(t) = \frac{1}{\gamma}\int_0^t \sinh\gamma(t-z)f(z)\,dz.$$

Solution: Two independent solutions of the homogeneous equation are given by

$$u_1(t) = e^{\gamma t} \qquad \text{and} \qquad u_2(t) = e^{-\gamma t},$$

and the Wronskian is

$$W(t) = \begin{vmatrix} e^{\gamma t} & e^{-\gamma t} \\ \gamma e^{\gamma t} & -\gamma e^{-\gamma t} \end{vmatrix} = -2\gamma.$$

We look for a particular solution of the form

$$u_p(t) = v_1(t)u_1(t) + v_2(t)u_2(t),$$

and forcing this to be a solution to the original differential equation gives

$$v_1'(t) = -\frac{u_2 f}{W} \qquad \text{and} \qquad v_2'(t) = \frac{u_1 f}{W},$$

and integrating, we get

$$v_1(t) = -\int_0^t \frac{u_2(z)f(z)}{W(z)}\,dz \qquad \text{and} \qquad v_2(t) = \int_0^t \frac{u_1(z)f(z)}{W(z)}\,dz.$$

Therefore the particular solution can be written as

$$u_p(t) = -u_1(t)\int_0^t \frac{u_2(z)f(z)}{W(z)}\,dz + u_2(t)\int_0^t \frac{u_1(z)f(z)}{W(z)}\,dz,$$

and so

$$u_p(t) = -\frac{1}{2\gamma}\left[-e^{\gamma t}\int_0^t e^{-\gamma z}\,dz + e^{-\gamma t}\int_0^t e^{\gamma z}\,dz\right] = \frac{1}{\gamma}\int_0^t \sinh\gamma(t-z)\,dz.$$

Question 12.

Many differential equations are really Bessel's equation in disguised form. Consider the following equation:

$$x^2\frac{d^2u}{dx^2} + (2c+1)x\,\frac{du}{dx} + \left[a^2b^2x^{2b} + (c^2 - \mu^2b^2)\right]u = 0 \qquad (*)$$

where a, b, c, μ are constants. (μ is not an integer).

(a) Show that the change of variables defined by

$$s = ax^b, \qquad \text{and} \qquad w(s) = x^c u(x)$$

transforms equation $(*)$ into Bessel's equation for $w(s)$.

(b) Write the general solution of the equation $(*)$ in terms of Bessel functions.

SOLUTION: Let $s = ax^b$, and $w(s) = x^c u(x)$, then

$$x = \left(\frac{s}{a}\right)^{\frac{1}{b}} \qquad \text{and} \qquad u(x) = \frac{w(s)}{\left(\frac{s}{a}\right)^{\frac{c}{b}}}.$$

Now,

$$\frac{ds}{dx} = abx^{b-1} = ab\left(\frac{s}{a}\right)^{\frac{b-1}{b}} = a^{\frac{1}{b}} b s^{1-\frac{1}{b}}.$$

Define

$$A = a^{\frac{c}{b}}, \quad \lambda = \frac{1}{b}, \quad B = a^{\frac{1}{b}} b,$$

then

$$\frac{ds}{dx} = Bs^{1-\lambda}, \quad u(x) = As^{-c\lambda} w(s),$$

and

$$\frac{du}{dx} = \frac{d}{ds}\big(As^{-c\lambda} w(s)\big)\,\frac{ds}{dx} = AB\big[s^{1-(1+c)\lambda} w' - c\lambda s^{-(1+c)\lambda} w\big]$$

and

$$\begin{aligned}\frac{d^2u}{dx^2} &= \frac{d}{ds}\big\{AB\big[s^{1-(1+c)\lambda} w' - c\lambda s^{-(1+c)\lambda} w\big]\big\}\,\frac{ds}{dx} \\ &= AB^2\big\{s^{2-(2+c)\lambda} w'' + [1-(1+2c)\lambda] s^{1-(2+c)\lambda} w' + c(1+c)\lambda^2 s^{-(2+c)\lambda} w\big\}\end{aligned}$$

and

$$\begin{aligned}x^2\,\frac{d^2u}{dx^2} &= a^{c\lambda} b^2 s^{-c\lambda}\big[s^2 w'' + [1-(1+2c)\lambda] s w' + c(1+c)\lambda^2 w\big] \\ (2c+1)x\frac{du}{dx} &= (1+2c)a^{c\lambda} b s^{-c\lambda}(sw' - c\lambda w)\end{aligned}$$

(check these!)

From $(*)$ we have

$$\begin{aligned}&a^{c\lambda} b^2 s^{-c\lambda}\{s^2 w'' + [1-(1+2c)\lambda] s w' + c(1+c)\lambda^2 w\} \\ &\qquad + a^{c\lambda} b s^{-c\lambda}(1+2c)(sw' - c\lambda w) + [a^2 b^2 a^{-2} s^2 + c^2 - \mu^2 b^2] a^{c\lambda} s^{-c\lambda} w = 0\end{aligned}$$

or

$$\begin{aligned}&a^{c\lambda} b^2 s^{-c\lambda}\Bigg\{s^2 w'' + \left[1-(1+2c)\lambda + \frac{1}{b}(1+2c)\right] s w' \\ &\qquad + \left[c(c+1)\lambda^2 - \frac{1}{b}(1+2c)c\lambda + s^2 + \frac{c^2}{b^2} - \mu^2\right] w\Bigg\} = 0.\end{aligned}$$

Therefore

$$s^2 w'' + s w' + (s^2 - \mu^2) w = 0$$

which is Bessel's equation, and the solution is

$$w(s) = c_1 J_\mu(s) + c_2 J_{-\mu}(s)$$

and

$$u(x) = x^{-c} w(s) = x^{-c} w(ax^b) = x^{-c}[c_1 J_\mu(ax^b) + c_2 J_{-\mu}(ax^b)].$$

Question 13.

Use the result of the previous problem to obtain the general solution of Airy's equation

$$u'' + xu = 0.$$

SOLUTION: Solve $u'' + xu = 0$ by choosing a, b, c, μ so that $(*)$ looks like $u'' + xu = 0$.

Since

$$\frac{d^2u}{dx^2} + (2c+1)\frac{1}{x}\frac{du}{dx} + \left[a^2b^2x^{2(b-1)} + (c^2 - \mu^2b^2)x^{-2}\right]u = 0,$$

we need

$$\begin{aligned} 2c + 1 &= 0 \\ 2(b-1) &= 1 \\ a^2b^2 &= 1 \\ c^2 - \mu^2b^2 &= 0, \end{aligned}$$

that is,

$$c = -\frac{1}{2}, \qquad b = \frac{3}{2}, \qquad a = \frac{2}{3}, \qquad \mu = \frac{1}{3}.$$

The solution to Airy's equation $u'' + xu = 0$ is

$$u(x) = x^{1/2}\left[c_1 J_{\frac{1}{3}}\left(\tfrac{2}{3}x^{\frac{3}{2}}\right) + c_2 J_{-\frac{1}{3}}\left(\tfrac{2}{3}x^{\frac{3}{2}}\right)\right].$$