Math 300 Spring - Summer 2018

Problem Set # 4

Question 1.

Solve the vibrating circular membrane problem for the radially symmetric case, that is, solve the initial value – boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{4}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad 0 \le r \le 1,\ t \ge 0$$

$$u(1,t) = 0,$$

$$u(r,0) = 5J_0\left(z_3\, r\right)$$

$$\frac{\partial u}{\partial t}(r,0) = 0, \quad 0 \le r \le 1,$$

where $J_0(z)$ denotes the Bessel function of the first kind of order zero, and z_n denotes the n^{th} zero of $J_0(z)$.

Question 2.

Solve the heat equation with time-dependent sources and boundary conditions:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$u(x,0) = f(x)$$

Reduce the problem to one with homogeneous boundary conditions if

$$u(0,t) = A(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = B(t).$$

Hint: First use $u(x,t) = w(x,t) + v(x,t)$ and assume that v satisfies just the boundary conditions (and nothing else), then use the method of eigenfunction expansions to solve for $w(x,t)$.

Question 3.

Solve the two-dimensional heat equation with circularly symmetric time-independent sources, boundary conditions, and initial conditions (inside a circle):

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with

$$u(r,0) = f(r) \qquad \text{and} \qquad u(a,t) = T.$$

Question 4.

Consider the heat equation with a steady source

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$

subject to the initial and boundary conditions:

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad \text{and} \qquad u(x,0) = f(x).$$

Obtain the solution by the method of eigenfunction expansion. Show that the solution approaches a steady-state solution.

Question 5.

(a) Show that the Fourier transform is a linear operator; that is, show that

$$\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] = c_1 F(\omega) + c_2 G(\omega)$$

(b) Show that $\mathcal{F}\left[f(x)g(x)\right] \neq F(\omega)G(\omega)$.

Question 6.

If $F(\omega)$ is the Fourier transform of $f(x)$, show that the inverse Fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$. This result is known as the **Shift Theorem** for Fourier transforms.

Question 7.

(a) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u, \quad -\infty < x < \infty,\ t > 0$$

$$u(x,0) = f(x), \quad -\infty < x < \infty.$$

(b) Does your solution suggest a simplifying transformation ?

Question 8.

Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty,\ t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < \infty.$$