Math 300 Spring-Summer 2018

Advanced Boundary Value Problems I

Problem Set # 1

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Note: Doing these problems on this problem set is good practice for the first midterm examination.

Question 1.

For the following functions, sketch the Fourier series of $f(x)$ on the interval $-L \leq x \leq L$, and determine the Fourier coefficients:

(a) $f(x) = \begin{cases} 1 & \text{for} \quad |x| < L/2 \\ 0 & \text{for} \quad |x| > L/2 \end{cases}$

(b) $f(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{if } -L < x < 0 \end{cases}$

Question 2.

Show that the Fourier series operation is linear: that is, show that the Fourier series of

$$c_1 f(x) + c_2 g(x)$$

is the sum of c_1 times the Fourier series of $f(x)$ and c_2 times the Fourier series of $g(x)$.

Question 3.

For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$, and determine the Fourier coefficients:

(a) $f(x) = \begin{cases} x & -L < x < 0 \\ 1 + x & 0 < x < L \end{cases}$

(b) $f(x) = \begin{cases} 2, & -L < x < 0 \\ e^{-x} & 0 < x < L \end{cases}$

Question 4.

Show that e^x is the sum of an even function and an odd function.

Question 5.

Find all solutions to the boundary value problem

$$\begin{aligned} \phi''(x) + \phi(x) &= 0, \quad 0 \leq x \leq 1 \\ \phi(0) &= 0 \\ \phi(1) &= 0. \end{aligned}$$

Question 6.

Consider the integral $\displaystyle\int_0^1 \frac{dx}{1+x^2}$.

(a) Evaluate the integral explicitly.

(b) Use the Taylor series of $\dfrac{1}{1+x^2}$ (a geometric series) to obtain an infinite series for the integral.

(c) Equate part (a) to part (b) in order to derive a formula for π.

Question 7.

For continuous functions,

(a) Under what conditions does $f(x)$ equal its Fourier series for all $x \in [-L, L]$?

(b) Under what conditions does $f(x)$ equal its Fourier sine series for all $x \in [0, L]$?

(c) Under what conditions does $f(x)$ equal its Fourier cosine series for all $x \in [0, L]$?

Hint: What does the Fourier series converge to at the end points of the interval?

Question 8.

Consider the boundary value – initial value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0; \qquad \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0; \qquad u(x,0) = f(x), \quad 0 < x < L.$$

Solve this problem by looking for a solution as a Fourier cosine series. Assume that u and $\dfrac{\partial u}{\partial x}$ are continuous and $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial u}{\partial t}$ are piecewise smooth. Justify all differentiations of infinite series.

Question 9. Solve Laplace's equation inside a rectangle $0 \le x \le L,\ 0 \le y \le H$, with the following boundary conditions:

(a) $\dfrac{\partial u}{\partial x}(0,y) = g(y), \qquad \dfrac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = 0, \qquad u(x,H) = 0$

(b) $\dfrac{\partial u}{\partial x}(0,y) = 0, \qquad \dfrac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = \begin{cases} 1 & \text{for } 0 < x < L/2, \\ 0 & \text{for } L/2 < x < L \end{cases} \qquad \dfrac{\partial u}{\partial y}(x,H) = 0$

Question 10. Solve Laplace's equation inside a circular annulus $(a < r < b)$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad a < r < b,\ -\pi < \theta < \pi$$

subject to the boundary conditions

$$\frac{\partial u}{\partial r}(a,\theta) = f(\theta), \qquad \frac{\partial u}{\partial r}(b,\theta) = g(\theta),$$

for $-\pi < \theta < \pi$.