Problem Set 0
Math 300 - Spring-Summer 2018

These problems are a review of the techniques used last year in Math 201.

Question 1.

Find the general solution to the differential equation

$$(e^x\varphi')' + \lambda^2 e^x \varphi = 0.$$

Question 2.

Compare and contrast the form of the solutions of these three differential equations and their behavior as $t \to \infty$.

(a) $\dfrac{d^2u}{dt^2} + u = 0$ (b) $\dfrac{d^2u}{dt^2} = 0$ (c) $\dfrac{d^2u}{dt^2} - u = 0.$

Question 3.

Find the general solution to the following differential equation (λ is a constant). Use an "exponential" trial solution.

$$\frac{d^4u}{dx^4} - \lambda^4 u = 0.$$

Question 4.

One solution of the differential equation

$$\frac{d}{dx}\left(x\,\frac{du}{dx}\right) + \frac{4x^2-1}{4x}\,u = 0$$

is given by

$$u_1(x) = \frac{\cos x}{\sqrt{x}}.$$

Find a second independent solution.

Question 5.

Given the differential equation

$$\frac{d}{d\rho}\left(\rho\,\frac{d\varphi}{d\rho}\right) + \frac{4\lambda^2\rho^2-1}{4\rho}\,\varphi = 0,$$

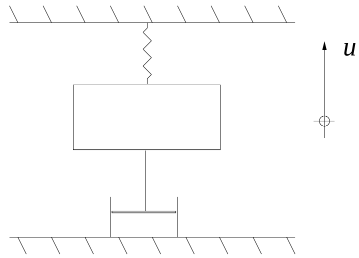
use the change of variable

$$\varphi(\rho) = \frac{v(\rho)}{\sqrt{\rho}}$$

to solve the equation.

Question 6.

The displacement $u(t)$ of a mass in mass-spring-damper system, as in the figure below,



is described by the initial value problem

$$\frac{d^2u}{dt^2} + b\frac{du}{dt} + \omega^2 u = 0$$
$$u(0) = u_0$$
$$\frac{du}{dt}(0) = v_0.$$

(The coefficients b and ω^2 are proportional to the characteristic constants of the damper and the spring, respectively.)

Solve the initial value problem for each of the parameter ranges below, and explain why these ranges might have been chosen.

(a) $b = 0$, (b) $0 < b < 2\omega$, (c) $b = 2\omega$, (d) $b > 2\omega$.

Question 7.

Find the general solution to the differential equation

$$\frac{d^2u}{dx^2} - \gamma^2(u - U) = 0$$

where U and γ^2 are constant.

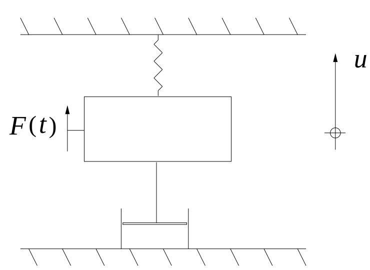
Question 8.

Find the general solution of the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -1.$$

Question 9.

The displacement $u(t)$ of a mass in mass-spring-damper system with an external force, as in the figure below,



is described by the initial value problem

$$\frac{d^2u}{dt^2} + b\frac{du}{dt} + \omega^2 u = f_0 \cos \mu t$$
$$u(0) = 0$$
$$\frac{du}{dt}(0) = 0.$$

(The coefficients b and ω^2 are proportional to the characteristic constants of the damper and the spring, respectively, and the coefficient f_0 is proportional to the amplitude of the external force.)

Solve the initial value problem for these three cases;

(a) $b = 0, \quad \mu \neq \omega,$ (b) $b = 0, \quad \mu = \omega,$ (c) $b > 0.$

Question 10.

Use variation of parameters to find a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \sin x,$$

if two independent solutions to the homogeneous equation are given by $y_1(x) = \cos x$, and $y_2(x) = \sin x$. Be sure that the differential equation is in the correct form.

Question 11.

Use variation of parameters to show that a particular solution of the differential equation

$$\frac{d^2u}{dt^2} - \gamma^2 u = f(t)$$

is given by

$$u_p(t) = \frac{1}{\gamma}\int_0^t \sinh \gamma(t-z) f(z)\, dz.$$

Question 12.

Many differential equations are really Bessel's equation in disguised form. Consider the following equation:

$$x^2\frac{d^2u}{dx^2} + (2c+1)x\,\frac{du}{dx} + \left[a^2b^2x^{2b} + (c^2 - \mu^2b^2)\right]u = 0 \qquad (*)$$

where a, b, c, μ are constants. (μ is not an integer).

(a) Show that the change of variables defined by

$$s = ax^b, \qquad \text{and} \qquad w(s) = x^c u(x)$$

transforms equation $(*)$ into Bessel's equation for $w(s)$.

(b) Write the general solution of the equation $(*)$ in terms of Bessel functions.

Question 13.

Use the result of the previous problem to obtain the general solution of Airy's equation

$$u'' + xu = 0.$$