**Solutions to Problem Set1**

**Math 300 - Spring-Summer 2018**

**Question 1.**

For the following functions, sketch the Fourier series of $f(x)$ on the interval $-L \le x \le L$, and determine the Fourier coefficients:

(a) $f(x) = \begin{cases} 1 & \text{for} \quad |x| < L/2 \\ 0 & \text{for} \quad |x| > L/2 \end{cases}$

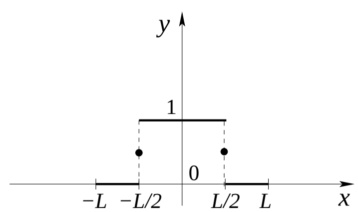
(b) $f(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{if } -L < x < 0 \end{cases}$

Solution:

(a) From Dirichlet's theorem the Fourier series of $f(x)$ converges to

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

if $-L \le x \le L$, and the graph of the Fourier series of $f(x)$ on the interval $-L \le x \le L$ is shown below.




Since $f(x)$ is an even piecewise smooth function on the interval $[-L, L]$, it has a Fourier series representation of the form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^{\frac{L}{2}} 1\,dx = \frac{1}{2},$$

and

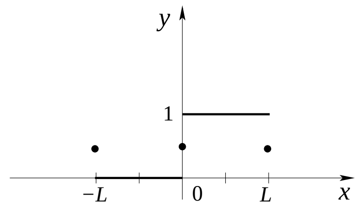
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^{\frac{L}{2}} \cos\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}\sin\frac{n\pi x}{L}\bigg|_0^{\frac{L}{2}} = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

for $n \ge 1$.

(b) Again, from Dirichlet's theorem the Fourier series of $f(x)$ converges to

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

if $-L \le x \le L$, and the graph of the Fourier series of $f(x)$ on the interval $-L \le x \le L$ is shown below.




Since $f(x) - \dfrac{1}{2}$ is an odd piecewise smooth function on the interval $[-L, L]$, it has a Fourier series representation of the form

$$f(x) - \frac{1}{2} \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L \left( f(x) - \frac{1}{2} \right) \sin \frac{n\pi x}{L} \, dx = \frac{1}{L} \int_0^L \sin \frac{n\pi x}{L} \, dx \\ &= -\frac{1}{n\pi} \cos \frac{n\pi x}{L} \bigg|_0^L = -\frac{1}{n\pi} (\cos n\pi - 1) \\ &= \frac{1}{n\pi} [1 - (-1)^n] \end{aligned}$$

for $n \geq 1$.

**Question 2.**

Show that the Fourier series operation is linear: that is, show that the Fourier series of

$$c_1 f(x) + c_2 g(x)$$

is the sum of $c_1$ times the Fourier series of $f(x)$ and $c_2$ times the Fourier series of $g(x)$.

Solution: Suppose that the Fourier series of $f$ and $g$ are given by

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \qquad \text{and} \qquad g(x) \sim C_0 + \sum_{n=1}^{\infty} \left( C_n \cos \frac{n\pi x}{L} + D_n \sin \frac{n\pi x}{L} \right)$$

where

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx, \quad A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

for $n \geq 1$, and

$$C_0 = \frac{1}{2L} \int_{-L}^{L} g(x) \, dx, \quad C_n = \frac{1}{L} \int_{-L}^{L} g(x) \cos \frac{n\pi x}{L} \, dx, \quad D_n = \frac{1}{L} \int_{-L}^{L} g(x) \sin \frac{n\pi x}{L} \, dx$$

for $n \geq 1$.

If $c_1$ and $c_2$ are scalars, and the Fourier series of $c_1 f + c_2 g$ is

$$c_1 f(x) + c_2 g(x) \sim E_0 + \sum_{n=1}^{\infty} \left( E_n \cos \frac{n\pi x}{L} + F_n \sin \frac{n\pi x}{L} \right),$$

then

$$E_0 = \frac{1}{2L} \int_{-L}^{L} (c_1 f(x) + c_2 g(x))\, dx = \frac{c_1}{2L} \int_{-L}^{L} f(x)\, dx + \frac{c_2}{2L} \int_{-L}^{L} g(x)\, dx = c_1 A_0 + c_2 C_0.$$

Also,

$$\begin{aligned} E_n &= \frac{1}{L} \int_{-L}^{L} (c_1 f(x) + c_2 g(x)) \cos \frac{n\pi x}{L}\, dx \\ &= \frac{c_1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx + \frac{c_2}{L} \int_{-L}^{L} g(x) \cos \frac{n\pi x}{L}\, dx \\ &= c_1 A_n + c_2 C_n \end{aligned}$$

for $n \geq 1$. Similarly,

$$\begin{aligned} F_n &= \frac{1}{L} \int_{-L}^{L} (c_1 f(x) + c_2 g(x)) \sin \frac{n\pi x}{L}\, dx \\ &= \frac{c_1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx + \frac{c_2}{L} \int_{-L}^{L} g(x) \sin \frac{n\pi x}{L}\, dx \\ &= c_1 B_n + c_2 D_n \end{aligned}$$

for $n \geq 1$. Therefore the Fourier series for $c_1 f + c_2 g$ is

$$\begin{aligned} c_1 f(x) + c_2 g(x) &\sim c_1 A_0 + c_2 C_0 + \sum_{n=1}^{\infty} \left( (c_1 A_n + c_2 C_n) \cos \frac{n\pi x}{L} + (c_1 B_n + c_2 D_n) \sin \frac{n\pi x}{L} \right) \\ &= c_1 \left[ A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \right] + c_2 \left[ C_0 + \sum_{n=1}^{\infty} \left( C_n \cos \frac{n\pi x}{L} + D_n \sin \frac{n\pi x}{L} \right) \right] \\ &\sim c_1 f(x) + c_2 g(x). \end{aligned}$$

**Question 3.**

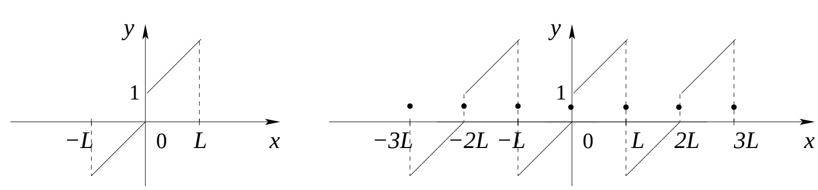
For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$, and determine the Fourier coefficients:

(a) $f(x) = \begin{cases} x & -L < x < 0 \\ 1 + x & 0 < x < L \end{cases}$

(b) $f(x) = \begin{cases} 2, & -L < x < 0 \\ e^{-x} & 0 < x < L \end{cases}$

SOLUTION:

(a) *Fourier Series*: The graphs of $f(x)$ and the Fourier series of $f(x)$ are shown below.



The Fourier series representation of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \frac{1}{2L} \int_{-L}^{L} x\, dx + \frac{1}{2L} \int_{0}^{L} 1\, dx = \frac{1}{2L} L = \frac{1}{2},$$

since the function $x$ is an odd function on $[-L, L]$.

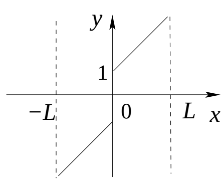
For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \tfrac{n\pi x}{L}\, dx = \frac{1}{L} \int_{-L}^{L} x \cos \tfrac{n\pi x}{L}\, dx + \frac{1}{L} \int_{0}^{L} \cos \tfrac{n\pi x}{L}\, dx \\ &= \frac{1}{n\pi} \sin \tfrac{n\pi x}{L} \bigg|_0^L = 0, \end{aligned}$$

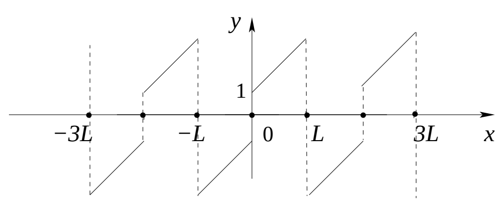
since $x \cos \frac{n\pi x}{L}$ is an odd function on $[-L, L]$, and

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \tfrac{n\pi x}{L}\, dx = \frac{1}{L} \int_{-L}^{L} x \sin \tfrac{n\pi x}{L}\, dx + \frac{1}{L} \int_{0}^{L} \sin \tfrac{n\pi x}{L}\, dx \\ &= \frac{2}{L} \int_{0}^{L} x \sin \tfrac{n\pi x}{L}\, dx + \frac{1}{L} \int_{0}^{L} \sin \tfrac{n\pi x}{L}\, dx \\ &= \frac{2}{L} \left[ -\frac{L}{n\pi} x \cos \tfrac{n\pi x}{L} \bigg|_0^L + \frac{L}{n\pi} \int_0^L \cos \tfrac{n\pi x}{L}\, dx \right] + \frac{1}{L} \left( -\frac{L}{n\pi} \cos \tfrac{n\pi x}{L} \right) \bigg|_0^L \\ &= \frac{1}{n\pi} \left[ 1 - (-1)^n \right] - \frac{2L(-1)^n}{n\pi}. \end{aligned}$$

*Fourier Sine Series*: The graphs of the odd extension of $f(x)$ to the interval $[-L, L]$ and the Fourier sine series of $f(x)$ are shown below.



*Odd Extension*



*Sine Series*

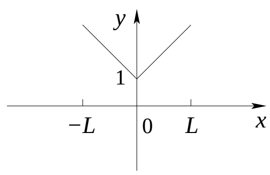
The Fourier sine series representation of $f(x)$ is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L},$$

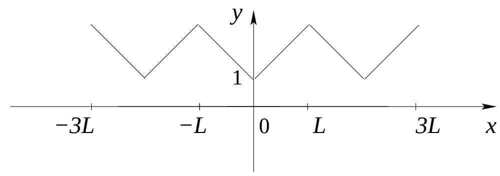
where

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi x}{L}\, dx = \frac{2}{L} \int_0^L (1 + x) \sin \tfrac{n\pi x}{L}\, dx \\ &= \frac{2}{L} \left[ -\frac{L}{n\pi} (1 + x) \cos \tfrac{n\pi x}{L} \bigg|_0^L + \frac{L}{n\pi} \int_0^L \cos \tfrac{n\pi x}{L}\, dx \right] \\ &= \frac{2}{n\pi} \left[ 1 - (-1)^n \right] - \frac{2L(-1)^n}{n\pi}. \end{aligned}$$

*Fourier Cosine Series*: The graphs of the even extension of $f(x)$ to the interval $[-L, L]$ and the Fourier cosine series of $f(x)$ are shown below.




Even Extension




Cosine Series

The Fourier cosine series representation of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L},$$

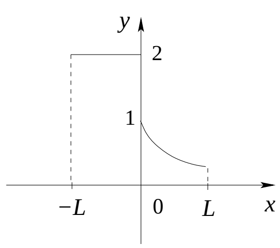
where

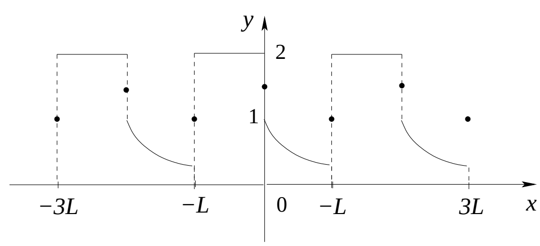
$$a_0 = \frac{1}{L} \int_0^L f(x)\,dx = \frac{1}{L} \int_0^L (1+x)\,dx = \frac{1}{L} \left[ x\Big|_0^L + \frac{x^2}{2}\Big|_0^L \right] = \left( 1 + \frac{L}{2} \right),$$

and for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \tfrac{n\pi x}{L}\,dx = \frac{2}{L} \int_0^L (1+x) \cos \tfrac{n\pi x}{L}\,dx \\ &= \frac{2}{L} \left[ \frac{L}{n\pi}(1+x) \sin \tfrac{n\pi x}{L} \Big|_0^L - \frac{L}{n\pi} \int_0^L \sin \tfrac{n\pi x}{L}\,dx \right] \\ &= \frac{2L}{n^2\pi^2} \cos \tfrac{n\pi x}{L} \Big|_0^L = \frac{2L}{n^2\pi^2} \left[ (-1)^n - 1 \right]. \end{aligned}$$

(b) *Fourier Series*: The graphs of $f(x)$ and the Fourier series of $f(x)$ are shown below.




The Fourier series representation of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx = \frac{1}{2L} \int_{-L}^{0} 2\,dx + \frac{1}{2L} \int_0^L e^{-x}\,dx = \frac{1}{2L} \left( 2L + 1 - e^{-L} \right).$$

Since

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right),$$

then

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\tfrac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} 2\cos\tfrac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} e^{-x}\cos\tfrac{n\pi x}{L}\,dx \\
&= \frac{2}{n\pi}\sin\tfrac{n\pi x}{L}\Big|_{-L}^{0} + \frac{n\pi}{L^2+n^2\pi^2}e^{-x}\sin\tfrac{n\pi x}{L}\Big|_{0}^{L} - \frac{L}{L^2+n^2\pi^2}e^{-x}\cos\tfrac{n\pi x}{L}\Big|_{0}^{L} \\
&= \frac{L}{L^2+n^2\pi^2}\left[1 - e^{-L}(-1)^n\right]
\end{aligned}$$

for $n \geq 1$.

Since

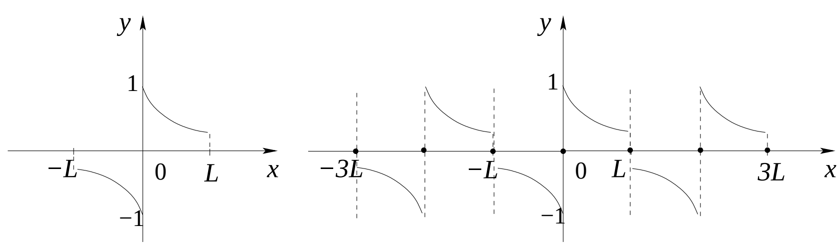
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right),$$

then

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\tfrac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} 2\sin\tfrac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} e^{-x}\sin\tfrac{n\pi x}{L}\,dx \\
&= -\frac{2}{n\pi}\cos\tfrac{n\pi x}{L}\Big|_{-L}^{0} - \frac{L}{L^2+n^2\pi^2}e^{-x}\sin\tfrac{n\pi x}{L}\Big|_{0}^{L} - \frac{n\pi}{L^2+n^2\pi^2}e^{-x}\cos\tfrac{n\pi x}{L}\Big|_{0}^{L} \\
&= \frac{n\pi}{L^2+n^2\pi^2}\left[1 - e^{-L}(-1)^n\right]
\end{aligned}$$

for $n \geq 1$.

*Fourier Sine Series*: The graphs of the odd extension of $f(x)$ to the interval $[-L, L]$ and the Fourier sine series of $f(x)$ are shown below.




The Fourier sine series representation of $f(x)$ is

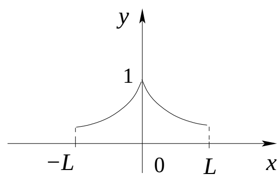
$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\tfrac{n\pi x}{L},$$

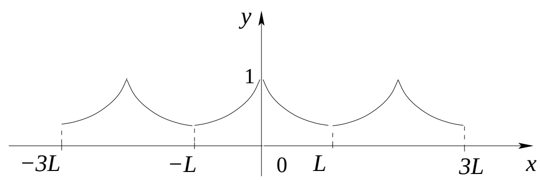
where

$$\begin{aligned}
b_n &= \frac{2}{L}\int_{0}^{L} f(x)\sin\tfrac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} e^{-x}\sin\tfrac{n\pi x}{L}\,dx \\
&= -\frac{2L}{L^2+n^2\pi^2}e^{-x}\sin\tfrac{n\pi x}{L}\Big|_{0}^{L} - \frac{2n\pi}{L^2+n^2\pi^2}e^{-x}\cos\tfrac{n\pi x}{L}\Big|_{0}^{L} \\
&= \frac{2n\pi}{L^2+n^2\pi^2}\left[1 - e^{-L}(-1)^n\right]
\end{aligned}$$

for $n \geq 1$.

*Fourier Cosine Series*: The graphs of the even extension of $f(x)$ to the interval $[-L, L]$ and the Fourier cosine series of $f(x)$ are shown below.




The Fourier cosine series representation of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L},$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L e^{-x}\,dx = -\frac{1}{L}e^{-x}\Big|_0^L = \frac{1}{L}\left(1 - e^{-L}\right).$$

Since

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right),$$

then

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos \tfrac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L e^{-x}\cos \tfrac{n\pi x}{L}\,dx \\ &= \frac{2n\pi}{L^2+n^2\pi^2}e^{-x}\sin \tfrac{n\pi x}{L}\Big|_0^L - \frac{2L}{L^2+n^2\pi^2}e^{-x}\cos \tfrac{n\pi x}{L}\Big|_0^L \\ &= \frac{2L}{L^2+n^2\pi^2}\left[1 - e^{-L}(-1)^n\right] \end{aligned}$$

for $n \geq 1$.

**Question 4.**

Show that $e^x$ is the sum of an even function and an odd function.

Solution: We can write

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cosh x + \sinh x,$$

and $\cosh x$ is an even function while $\sinh x$ is an odd function.

In general, if $f(x)$ is an arbitrary function, then we can write

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

where

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

is even, and

$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$

is odd.

**Question 5.**

Find all solutions to the boundary value problem

$$\begin{aligned} \phi''(x) + \phi(x) &= 0, \quad 0 \le x \le 1 \\ \phi(0) &= 0 \\ \phi(1) &= 0. \end{aligned}$$

SOLUTION: The general solution to the differential equation is

$$\phi(x) = A\cos x + B\sin x, \quad 0 \le x \le 1.$$

Appying the first boundary condition,

$$\phi(0) = A = 0,$$

the solution becomes

$$\phi(x) = B\sin x, \quad 0 \le x \le 1.$$

Applying the second boundary condition,

$$\phi(1) = B\sin 1 = 0,$$

and since $\sin 1 \ne 0$, then $B = 0$, and the only solution to the boundary value problem is the trivial solution

$$\phi(x) = 0$$

for $0 \le x \le 1$.

**Question 6.**

Consider the integral $\displaystyle\int_0^1 \frac{dx}{1+x^2}$.

(a) Evaluate the integral explicitly.

(b) Use the Taylor series of $\dfrac{1}{1+x^2}$ (a geometric series) to obtain an infinite series for the integral.

(c) Equate part (a) to part (b) in order to derive a formula for $\pi$.

SOLUTION:

(a) Since

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2},$$

we have

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\,\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

(b) Recall that the geometric series

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots,$$

that is,

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

converges for all $-1 < t < 1$.

Integrating from 0 to $x$, where $|x| < 1$, we get

$$\begin{aligned}
\int_0^x \frac{1}{1+t^2}\,dt &= \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n t^{2n}\right)\,dt \\
&= \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}\,dt \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{2n+1}\bigg|_0^x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1},
\end{aligned}$$

and therefore

$$\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

for $-1 < x < 1$, and this is *Gregory's series* for $\tan^{-1} x$, discovered by James Gregory about 1670. Letting $x \to 1^-$, then a theorem of Abel tells us that

$$\begin{aligned}
\frac{\pi}{4} &= \int_0^1 \frac{1}{1+t^2}\,dt = \lim_{x\to 1^-}\int_0^x \frac{1}{1+t^2}\,dt = \lim_{x\to 1^-}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\lim_{x\to 1^-}x^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1},
\end{aligned}$$

so that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots,$$

this is *Leibniz's formula* for $\dfrac{\pi}{4}$, discovered by Leibniz in 1673.

(c) From part (b), we have

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

The convergence is very slow however.

Another proof of Leibniz's formula which doesn't require integrating an infinite series term-by-term is given below.

$$\begin{aligned}
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1} &= \int_0^1 \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2}\right)\,dx \\
&= \int_0^1 \frac{1-x^{2n}}{1+x^2}\,dx \\
&= \int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x^{2n}}{1+x^2}\,dx
\end{aligned}$$

and therefore,

$$\left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n-1}}{2n-1}\right)\right| = \int_0^1 \frac{x^{2n}}{1+x^2}\,dx \le \int_0^1 x^{2n}\,dx = \frac{1}{2n+1} \longrightarrow 0$$

as $n \to \infty$.

**Question 7.**

For continuous functions,

(a) Under what conditions does $f(x)$ equal its Fourier series for all $x \in [-L, L]$?

(b) Under what conditions does $f(x)$ equal its Fourier sine series for all $x \in [0, L]$?

(c) Under what conditions does $f(x)$ equal its Fourier cosine series for all $x \in [0, L]$?

**Hint:** What does the Fourier series converge to at the end points of the interval?

(a) From Dirichlet's theorem, we know that for any $x_0$ with $-L < x_0 < L$, the Fourier series of $f$ converges to $f(x_0)$ since $f$ is continuous at $x_0$.

We also know that at the endpoints $x = -L$ and $x = L$, the Fourier series converges to

$$\frac{1}{2}\left[f(L^-) + f(-L^+)\right],$$

and if $f$ is continuous at the endpoints, that is, continuous from the left at $x = L$ and continuous from the right at $x = -L$, then the Fourier series converges to

$$\frac{f(L) + f(-L)}{2}$$

at $x = L$ and at $x = -L$, so that the Fourier series converges to $f(x)$ for all $x \in [-L, L]$ if and only if $f(L) = f(-L)$.

(b) Again, from Dirichlet's theorem, if $0 < x_0 < L$, then the Fourier sine series of $f$ converges to $f(x_0)$ since $f$ is continuous at $x_0$.

If $f_{\text{odd}}$ is the odd extension of $f$ to $[-L, L]$, then at $x = 0$, the Fourier sine series of $f$ converges to

$$\frac{1}{2}\left[f_{\text{odd}}(0^-) + f_{\text{odd}}(0^+)\right] = \frac{1}{2}\left[-f(0) + f(0)\right] = 0,$$

and the Fourier sine series converges to $f$ at $x = 0$ if and only if $f(0) = 0$.

If $f_{\text{odd}}$ is the odd extension of $f$ to $[-L, L]$, then at $x = L$, the Fourier sine series of $f$ converges to

$$\frac{1}{2}\left[f_{\text{odd}}(L^-) + f_{\text{odd}}(-L^+)\right] = \frac{1}{2}\left[f_{\text{odd}}(L) + f_{\text{odd}}(-L)\right] = \frac{1}{2}\left[f(L) - f(L)\right] = 0,$$

and the Fourier sine series converges to $f$ at $x = L$ if and only if $f(L) = 0$.

(c) From Dirichlet's theorem, if $0, x_0 < L$, then the Fourier cosine series of $f$ converges to $f(x_0)$ since $f$ is continuous at $x_0$.

If $f_{\text{even}}$ is the even extension of $f$ to $[-L, L]$, then at $x = 0$, the Fourier cosine series of $f$ converges to

$$\frac{1}{2}\left[f_{\text{even}}(0^-) + f_{\text{even}}(0^+)\right] = \frac{1}{2}\left[f(0) + f(0)\right] = f(0),$$

and the Fourier cosine series of $f$ converges to $f$ at $x = 0$ if and only if $f$ is continuous from the right at $x = 0$.

If $f_{\text{even}}$ is the even extension of $f$ to $[-L, L]$, then at $x = L$, the Fourier cosine series of $f$ converges to

$$\frac{1}{2}\left[f_{\text{even}}(L^-) + f_{\text{even}}(-L^+)\right] = \frac{1}{2}\left[f_{\text{even}}(L) + f_{\text{even}}(-L)\right] \frac{1}{2}\left[f(L) + f(L)\right] = f(L),$$

and the Fourier cosine series of $f$ converges to $f$ at $x = L$ if and only if $f$ is continuous from the left at $x = L$.

**Question 8.**

Consider the boundary value – initial value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0; \qquad \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0; \qquad u(x,0) = f(x), \quad 0 < x < L.$$

Solve this problem by looking for a solution as a Fourier cosine series. Assume that $u$ and $\dfrac{\partial u}{\partial x}$ are continuous and $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial u}{\partial t}$ are piecewise smooth. Justify all differentiations of infinite series.

Solution: We assume a solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos\frac{n\pi x}{L}$$

and assuming all derivatives are continuous, we have

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=0}^{\infty} a_n(t)\left(\frac{n\pi}{L}\right)^2 \cos\frac{n\pi x}{L}$$

and since $u(x,t)$ satisfies the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

then we have

$$\sum_{n=0}^{\infty} a_n'(t)\cos\frac{n\pi x}{L} = -k\sum_{n=0}^{\infty} a_n(t)\left(\frac{n\pi}{L}\right)^2 \cos\frac{n\pi x}{L}.$$

Collecting terms that multiply $\cos\frac{n\pi x}{L}$ for $n \geq 0$ for $n \geq 1$, and using the fact that these trigonometric functions are linearly independent (they are orthogonal on the interval $[0,L]$), then we get

$$a_n'(t) = -ka_n(t)\left(\frac{n\pi}{L}\right)^2,$$

and we can solve these first order linear ordinary differential equations for $a_n(t)$ to get

$$a_n(t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt},$$

and the solution $u(x,t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\cos\frac{n\pi x}{L}.$$

Differentiating this with respect to $x$, we get

$$\frac{\partial u}{\partial x}(x,t) = -\sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L},$$

and setting $x = 0$, we get

$$0 = \frac{\partial u}{\partial x}(0,t),$$

and the first boundary condition is satisfied.

The solution is now

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \cos\frac{n\pi x}{L},$$

and we note that the second boundary condition $\dfrac{\partial u}{\partial x}(L,t) = 0$ is also satisfied, so we only need to find the constants $A_n$ to satisfy the initial condition $u(x,0) = f(x)$.

Setting $t = 0$ in the above expression for $u(x,t)$, we have

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L},$$

and the $A_n$ are the Fourier cosine series coefficients of $f(x)$, so that

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n \geq 1$$

and for $n = 0$,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx.$$

**Question 9.** Solve Laplace's equation inside a rectangle $0 \leq x \leq L,\ 0 \leq y \leq H$, with the following boundary conditions:

(a) $\dfrac{\partial u}{\partial x}(0,y) = g(y), \qquad \dfrac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = 0, \qquad u(x,H) = 0$

(b) $\dfrac{\partial u}{\partial x}(0,y) = 0, \qquad \dfrac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = \begin{cases} 1 & \text{for} \quad 0 < x < L/2, \\ 0 & \text{for} \quad L/2 < x < L \end{cases} \qquad \dfrac{\partial u}{\partial y}(x,H) = 0$

Solution:

(a) We assume a solution of the form $u(x,y) = X(x)\cdot Y(y)$, and substituting this into Laplace's equation we have

$$X''(x)\cdot Y(y) + X(x)\cdot Y''(y) = 0,$$

and

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

so we get two ordinary differential equations

$$X''(x) - \lambda X(x) = 0 \qquad \text{and} \qquad Y''(y) + \lambda Y(y) = 0.$$

We can satisfy the (homogeneous) boundary conditions by requiring that

$$Y(0) = 0, \quad Y(H) = 0 \qquad \text{and} \qquad X'(L) = 0.$$

Therefore $X$ and $Y$ satisfy the boundary value problems

$$X''(x) - \lambda X(x) = 0, \quad 0 \leq x \leq L \qquad Y''(y) + \lambda Y(y) = 0, \quad 0 \leq y \leq H$$

$$X'(L) = 0 \qquad\qquad Y(0) = 0$$

$$Y(H) = 0.$$

We solve the complete (Dirichlet) boundary value problem for $Y$ first, the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2$$

with corresponding eigenfunctions

$$Y_n(y) = \sin\tfrac{n\pi}{H}y$$

for $n \geq 1$.

The corresponding functions $X(x)$ satisfy the boundary value problem

$$X_n'' - \lambda_n X_n = 0, \quad 0 < x < L$$
$$X_n'(L) = 0,$$

and since the boundary condition at $x = L$ is homogeneous, we choose the following representation of the general solution

$$X_n(x) = A \cosh \tfrac{n\pi}{H}(L - x) + B \sinh \tfrac{n\pi}{H}(L - x),$$

and the condition $X_n'(L) = 0$ implies that $B = 0$. Therefore the solution to the boundary value problem for $X$ is

$$X_n(x) = \cosh \tfrac{n\pi}{H}(L - x), \quad 0 < x < L$$

for $n \geq 1$.

From the superposition principle, the function

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \tfrac{n\pi}{H} y \cosh \tfrac{n\pi}{H}(L - x) \tag{$*$}$$

satisfies Laplace's equation in the region $0 < x < L$, $0 < y < H$, and satisfies all of the boundary conditions except $\dfrac{\partial u}{\partial x}(0, y) = g(y)$.

In order to satisfy this condition, we have to use the orthogonality of the eigenfunctions on the interval $0 \leq y \leq H$. Differentiating $(*)$ with respect to $x$, and setting $x = 0$ we get

$$g(y) = \frac{\partial u}{\partial x}(0, y) = -\sum_{n=1}^{\infty} \frac{n\pi}{H} B_n \sin \tfrac{n\pi}{H} y \sinh \tfrac{n\pi}{H} L,$$

multiply both sides of this equation by $\sin \frac{m\pi}{H} y$, and integrate over the interval $0 \leq y \leq H$, to get

$$\int_0^H g(y) \sin \tfrac{m\pi}{H} y \, dy = -\sum_{n=1}^{\infty} \frac{n\pi}{H} \sinh \tfrac{n\pi L}{H} B_n \int_0^H \sin \tfrac{m\pi}{H} y \sin \tfrac{n\pi}{H} y \, dy$$

and using the orthogonality of the eigenfunctions, we have

$$B_m = \frac{-2}{m\pi \sinh \frac{m\pi L}{H}} \int_0^H g(y) \sin \tfrac{m\pi}{H} y \, dy \tag{$**$}$$

for $m \geq 1$.

The solution to Laplace's equation satisfying the given boundary conditions is given by $(*)$, where the coefficients $B_m$, $m \geq 1$, are given by $(**)$.

(b) Assuming a solution of the form $u(x, y) = X(x) \cdot Y(y)$ and separating variables we get the boundary value problems

$$X''(x) + \lambda X(x) = 0, \quad 0 \leq x \leq L \qquad Y''(y) - \lambda Y(y) = 0, \quad 0 \leq y \leq H$$
$$X'(0) = 0 \qquad Y'(H) = 0$$
$$X'(L) = 0.$$

We solve the complete (Neumann) boundary value problem first, the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

with corresponding eigenfunctions

$$X_n(x) = \cos \tfrac{n\pi}{L} x$$

for $n \geq 0$.

The corresponding functions $Y_n(y)$ satisfy the boundary value problem

$$\begin{aligned} Y_n'' - \lambda_n Y_n &= 0, \quad 0 < y < H \\ Y_n'(H) &= 0, \end{aligned}$$

and since the boundary condition at $y = H$ is homogeneous, we choose to represent the general solution as follows

$$Y_n(y) = A \cosh \tfrac{n\pi}{L}(H - y) + B \sinh \tfrac{n\pi}{L}(H - y),$$

and now the condition $Y_n'(H) = 0$ implies that $B = 0$. Therefore the solution to the boundary value problem for $Y$ is

$$Y_n(y) = \cosh \tfrac{n\pi}{L}(H - y), \quad 0 < y < H$$

for $n \geq 0$.

From the superposition principle, the function

$$u(x, y) = \sum_{n=0}^{\infty} A_n \cos \tfrac{n\pi}{L} x \, \cosh \tfrac{n\pi}{L}(H - y) \tag{$*$}$$

satisfies Laplace's equation in the region $0 < x < L$, $0 < y < H$, and satisfies all of the boundary conditions except

$$u(x, 0) = f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < L/2, \\ 0 & \text{for} \quad L/2 < x < L. \end{cases}$$

In order to satisfy this condition, we have to use the orthogonality of the eigenfunctions on the interval $0 \leq x \leq L$. Setting $y = 0$ we get

$$f(x) = u(x, 0) = \sum_{n=0}^{\infty} A_n \cos \tfrac{n\pi}{L} x \, \cosh \tfrac{n\pi H}{L},$$

multiply both sides of this equation by $\cos \frac{m\pi}{L} x$, and integrate over the interval $0 \leq x \leq L$, to get

$$\int_0^L f(x) \cos \tfrac{m\pi}{L} x \, dx = \sum_{n=0}^{\infty} \cosh \tfrac{n\pi H}{L} A_n \int_0^L \cos \tfrac{m\pi}{L} x \, \cos \tfrac{n\pi}{L} x \, dx$$

so that

$$A_0 = \frac{1}{2} \qquad \text{and} \qquad A_m = \frac{2 \sin \frac{m\pi}{2}}{m\pi \cosh \frac{m\pi H}{L}} \tag{$**$}$$

for $m \geq 1$.

From $(*)$ and $(**)$ the solution to Laplace's equation satisfying the given boundary conditions is given by

$$u(x, y) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2 \sin \frac{n\pi}{2}}{n\pi \cosh \frac{n\pi H}{L}} \cos \tfrac{n\pi}{L} x \, \cosh \tfrac{n\pi}{L}(H - y)$$

for $0 < x < L$ and $0 < y < H$.

**Question 10.** Solve Laplace's equation inside a circular annulus ($0 < a < r < b$)

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad a < r < b,\ -\pi < \theta < \pi$$

subject to the boundary conditions

$$\frac{\partial u}{\partial r}(a,\theta) = f(\theta), \qquad \frac{\partial u}{\partial r}(b,\theta) = g(\theta),$$

for $-\pi < \theta < \pi$.

SOLUTION: Note that we need to include two periodicity conditions to get the right number of boundary conditions:

$$u(r,-\pi) = u(r,\pi) \qquad \text{and} \qquad \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

for $a \le r \le b$.

We assume a solution of the form $u(r,\theta) = \phi(\theta)\cdot G(r)$, and substitute this into Laplace's equation to get

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.$$

We can satisfy the periodicity conditions by requiring that

$$\phi(-\pi) = \phi(\pi) \qquad \text{and} \qquad \phi'(-\pi) = \phi'(\pi),$$

and we can satisfy the boundary condition $\dfrac{\partial u}{\partial r}(a,\theta) = 0$ by requiring $G'(a) = 0$. and we have two boundary value problems:

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = 0, \quad a < r < b \qquad \phi''(\theta) + \lambda\phi(\theta) = 0, \quad -\pi < \theta < \pi$$

$$G'(a) = 0 \qquad \phi(-\pi) = \phi(\pi)$$

$$\phi'(-\pi) = \phi'(\pi).$$

We solve the complete (two periodicity conditions) boundary value problem for $\phi$ first, again we consider three cases.

case (i): If $\lambda = 0$, the general solution to the differential equation $\phi'' = 0$ is $\phi(\theta) = A\theta + B$, with $\phi'(\theta) = A$. The first periodicity condition implies that

$$-A\pi + B = A\pi + B,$$

so that $A = 0$. The solution is now $\phi(\theta) = B$, and the second periodicity condition is also satisfied, the (nontrivial) solution is $\phi(\theta) = B$. In this case, the eigenvalue is $\lambda_0 = 0$ with corresponding eigenfunction $\phi_0(\theta) = 1$.

case (ii): If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' - \mu^2\phi = 0$ is

$$\phi(\theta) = A\cosh\mu\theta + B\sinh\mu\theta, \qquad \text{with} \qquad \phi'(\theta) = \mu A\sinh\mu\theta + \mu B\cosh\mu\theta.$$

The first periodicity condition implies that

$$A\cosh(-\mu\pi) + B\sinh(-\mu\pi) = A\cosh\mu\pi + B\mu\sinh\mu\pi,$$

and since $\cosh\mu\theta$ is an even function and $\sinh\mu\theta$ is an odd function, then

$$2B\sinh\mu\pi = 0,$$

so that $B = 0$. The solution is now $\phi(\theta) = A\cosh\mu\theta$, and the second periodicity condition implies that

$$\mu A\sinh(-\mu\pi) = \mu A\sinh\mu\pi,$$

so that $2\mu A\sinh\mu\pi = 0$, and so $A = 0$. In this case we have only the trivial solution $\phi(\theta) = 0, \ -\pi < \theta < \pi$.

case (iii): If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the general solution to the differential equation $\phi'' + \mu^2\phi = 0$ is

$$\phi(\theta) = A\cos\mu\theta + B\sin\mu\theta, \qquad \text{with} \qquad \phi'(\theta) = -\mu A\sin\mu\theta + \mu B\cos\mu\theta.$$

The first periodicity condition implies that

$$A\cos(-\mu\pi) + B\sin(-\mu\pi) = A\cos\mu\pi + B\mu\sin\mu\pi,$$

and since $\cos\mu\theta$ is an even function and $\sin\mu\theta$ is an odd function, then

$$2B\sin\mu\pi = 0.$$

The second periodicity condition implies that

$$-\mu A\sin(-\mu\pi) + \mu B\cos(-\mu\pi) = -\mu A\sin\mu\pi + \mu B\cos\mu\pi,$$

so that

$$2\mu A\sin\mu\pi = 0.$$

There is a nontrivial solution if and only if at least one of $A$ and $B$ is nonzero, and the above implies that $\sin\mu\pi = 0$, that is, $\mu\pi = n\pi$ for some integer $n$. In this case the eigenvalues are $\lambda_n = n^2$, with corresponding eigenfunctions

$$\phi_n(\theta) = \cos n\theta \qquad \text{and} \qquad \phi_n(\theta) = \sin n\theta$$

for $n \geq 1$.

If $n \geq 1$, and we assume a solution to the corresponding equation

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - n^2 G = 0$$

of the form $G(r) = r^\alpha$, then

$$r\frac{d}{dr}\left(\alpha r^\alpha\right) - n^2 r^\alpha = 0,$$

that is,

$$\alpha^2 r^\alpha - n^2 r^\alpha = 0,$$

so that $\alpha = \pm n$, and we get two linearly independent solutions

$$G_{1n}(r) = r^n \qquad \text{and} \qquad G_{2n}(r) = \frac{1}{r^n}$$

and the general solution is

$$G_n(r) = Ar^n + \frac{B}{r^n},$$

for $n \geq 1$.

If $n = 0$, the corresponding differential equation for $G(r)$ is

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0,$$

and we get two linearly independent solutions

$$G_{10}(r) = 1 \qquad \text{and} \qquad G_{20}(r) = \log r,$$

and the general solution is

$$G_0(r) = A + B\log r,$$

From the superposition principle, the function

$$u(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\left[r^n(A_n\cos n\theta + B_n\sin n\theta) + \frac{1}{r^n}(C_n\cos n\theta + D_n\sin n\theta)\right] \tag{†}$$

with

$$\frac{\partial u}{\partial r}(r,\theta) = \frac{B_0}{r} + \sum_{n=1}^{\infty}\left[nr^{n-1}(A_n\cos n\theta + B_n\sin n\theta) - \frac{n}{r^{n+1}}(C_n\cos n\theta + D_n\sin n\theta)\right]$$

satisfies the periodicity conditions and Laplace's equation in the annular region $a \le r \le b,\ -\pi \le \theta \le \pi$.

We can satisfy the boundary conditions

$$\frac{\partial u}{\partial r}(a,\theta) = f(\theta) \qquad \text{and} \qquad \frac{\partial u}{\partial r}(b,\theta) = g(\theta)$$

for ; $-\pi < \theta < \pi$ by requiring that

$$\begin{aligned} f(\theta) &= \frac{B_0}{a} + \sum_{n=1}^{\infty}\left[na^{n-1}(A_n\cos n\theta + B_n\sin n\theta) - \frac{n}{a^{n+1}}(C_n\cos n\theta + D_n\sin n\theta)\right] \\ g(\theta) &= \frac{B_0}{b} + \sum_{n=1}^{\infty}\left[nb^{n-1}(A_n\cos n\theta + B_n\sin n\theta) - \frac{n}{b^{n+1}}(C_n\cos n\theta + D_n\sin n\theta)\right] \end{aligned} \tag{††}$$

where the coefficients are determined using the orthogonality of the eigenfunctions

$$\{\,1,\ \cos\theta,\ \sin\theta,\ \cos 2\theta,\ \sin 2\theta,\ \cos 3\theta,\ \sin 3\theta,\ \cdots\}$$

on the interval $-\pi \le \theta \le \pi$.

Multiplying equations (††) above by the eigenfunction 1 and integrating over the interval $[-\pi,\pi]$, we obtain

$$B_0 = \frac{a}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \qquad \text{and} \qquad B_0 = \frac{b}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta,$$

that is,

$$\int_{-\pi}^{\pi} f(\theta)\,a d\theta = \int_{-\pi}^{\pi} g(\theta)\,b d\theta.$$

Note that this also follows from the divergence theorem, since

$$\int_{-\pi}^{\pi}\frac{\partial u}{\partial r}(b,\theta)\,b\,d\theta - \int_{-\pi}^{\pi}\frac{\partial u}{\partial r}(a,\theta)\,a\,d\theta = \int_{\partial D}\text{grad u}\cdot\mathbf{n}\,ds = \iint_D \Delta u\,r dr d\theta = 0,$$

where $D$ is the closed annular region between the circles $r = a$ and $r = b$ and $\mathbf{n}$ is the outward unit normal to the boundary of $D$.

Multiplying the equations (††) by the appropriate eigenfunctions and integrating over the interval $[-\pi, \pi]$, we get

$$\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta = n\pi \left( a^{n-1} A_n - \frac{1}{a^{n+1}} C_n \right)$$

$$\int_{-\pi}^{\pi} g(\theta) \cos n\theta \, d\theta = n\pi \left( b^{n-1} A_n - \frac{1}{b^{n+1}} C_n \right)$$

$$\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta = n\pi \left( a^{n-1} B_n - \frac{1}{a^{n+1}} D_n \right)$$

$$\int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta = n\pi \left( b^{n-1} B_n - \frac{1}{b^{n+1}} D_n \right),$$

and solving for $A_n$, $B_n$, $C_n$, and $D_n$, we have

$$A_n = \frac{1}{n\pi(b^{2n} - a^{2n})} \left[ b^n \int_{-\pi}^{\pi} g(\theta) \cos n\theta \, b \, d\theta - a^n \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, a \, d\theta \right]$$

$$B_n = \frac{1}{n\pi(b^{2n} - a^{2n})} \left[ b^n \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, b \, d\theta - a^n \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, a \, d\theta \right]$$

$$C_n = \frac{a^n b^n}{n\pi(b^{2n} - a^{2n})} \left[ a^n \int_{-\pi}^{\pi} g(\theta) \cos n\theta \, b \, d\theta - b^n \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, a \, d\theta \right]$$

$$D_n = \frac{a^n b^n}{n\pi(b^{2n} - a^{2n})} \left[ a^n \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, b \, d\theta - b^n \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, a \, d\theta \right]$$

for $n \geq 1$.

The solution to the Neumann problem for Laplace's equation in the annulus $a < r < b$ is given by (†), where the coefficients $A_n$, $B_n$, $C_n$, and $D_n$ for $n \geq 1$ are given above, while

$$B_0 = \frac{a}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta = \frac{b}{2\pi} \int_{-\pi}^{\pi} g(\theta) \, d\theta,$$

and $A_0$ is an arbitrary constant.