Solutions to Practice Problems for Final Examination

Question 1. Given the function

$$f(x) = x, \quad -\pi < x < \pi$$

find the Fourier series for f and use Dirichlet's convergence theorem to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin na}{n} = \frac{a}{2}$$

for $0 < a < \pi$.

SOLUTION: Since $f(x)$ is an odd function on the interval $[-\pi, \pi]$, the Fourier series of $f(x)$ is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx \\ &= \frac{2}{\pi} \left[-\frac{1}{n} x \cos nx \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right] \\ &= -\frac{2}{n}(-1)^n \end{aligned}$$

and

$$b_n = \frac{2(-1)^{n-1}}{n}$$

for $n \geq 1$.

Therefore

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n},$$

and from Dirichlet's convergence theorem, since $f(x)$ is continuous for $-\pi < x < \pi$, the Fourier series converges to $f(x)$ for $-\pi < x < \pi$, that is,

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n}$$

for $-\pi < x < \pi$, in particular, choosing $x = a$, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin na}{n} = \frac{a}{2}$$

for $0 < a < \pi$.

Question 2. Let $0 < a < \pi$, given the function

$$f(x) = \begin{cases} \dfrac{1}{2a} & \text{if} \quad |x| < a \\ 0 & \text{if} \quad x \in (-\pi, \pi], \quad \text{and} \quad |x| > a \end{cases}$$

find the Fourier series for f and use Dirichlet's convergence theorem to show that

$$\sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{1}{2}(\pi - a)$$

for $0 < a < \pi$.

Solution: Since $f(x)$ is an even function of the interval $[-\pi, \pi]$, the Fourier series of $f(x)$ is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{a} \frac{1}{2a}\, dx = \frac{1}{2\pi},$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx \\ &= \frac{2}{\pi} \int_0^{a} \frac{1}{2a} \cos nx\, dx \\ &= \frac{1}{\pi a} \int_0^{a} \cos nx\, dx \\ &= \frac{1}{\pi a} \cdot \frac{1}{n} \sin nx \bigg|_0^{a} \\ &= \frac{1}{\pi a} \cdot \frac{\sin na}{n}, \end{aligned}$$

that is,

$$a_n = \frac{1}{\pi a} \cdot \frac{\sin na}{n}$$

for $n \geq 1$, and

$$f(x) \sim \frac{1}{2\pi} + \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{\sin na \cos nx}{n}$$

for $-\pi < x < \pi$.

Since $f(x)$ is continuous on the interval $-\pi < x < \pi$ the Fourier series converges to $f(x)$ for $-\pi < x < \pi$, that is,

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{\sin na \cos nx}{n}$$

for $-\pi < x < \pi$, in particular, when $x = 0$, we have

$$\frac{1}{2a} = \frac{1}{2\pi} + \frac{1}{\pi a} \sum_{n=1}^{\infty} \frac{\sin na}{n},$$

so that

$$\sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{1}{2}(\pi - a)$$

for $0 < a < \pi$.

Question 3. Consider the regular Sturm-Liouville problem

$$(x\phi')' + \lambda^2 \frac{1}{x}\phi = 0 \qquad 1 \le x \le 2$$

$$\phi(1) = 0$$

$$\phi(2) = 0$$

(a) The general solution to the differential equation is

$$\phi(x) = A\cos(\lambda \ln x) + B\sin(\lambda \ln x).$$

Find the eigenvalues λ_n^2 and the corresponding eigenfunctions ϕ_n for this problem.

(b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(c) Use the Rayleigh quotient to estimate the smallest eigenvalue of this regular Sturm-Liouville problem.
Note: From part (a), the first eigenvalue and eigenfunction are

$$\lambda_1^2 = \left(\frac{\pi}{\ln 2}\right)^2 \approx 20.5423 \qquad \text{and} \qquad \phi_1(x) = \sin\left(\frac{\pi \ln x}{\ln 2}\right).$$

Try to find a reasonable estimate.

Solution:

(a) If

$$\phi(x) = A\cos(\lambda \ln x) + B\sin(\lambda \ln x)$$

for $1 < x < 2$, then

$$\phi'(x) = -\frac{\lambda A}{x}\sin(\lambda \ln x) + \frac{\lambda B}{x}\cos(\lambda \ln x),$$

so that

$$x\phi'(x) = -\lambda A \sin(\lambda \ln x) + \lambda B\cos(\lambda \ln x),$$

and

$$(x\phi'(x))' = -\frac{\lambda^2 A}{x}\cos(\lambda \ln x) - \frac{\lambda^2 B}{x}\sin(\lambda \ln x).$$

Therefore

$$(x\phi'(x))' + \lambda^2 \frac{1}{x}\phi(x) = 0$$

for $1 < x < 2$, and $\phi(x)$ is a solution to the differential equation.
In order to satisfy the first boundary condition $\phi(1) = 0$, we need

$$\phi(1) = A\cos 0 + B\sin 0 = A = 0,$$

and the solution is now

$$\phi(x) = B\sin(\lambda \ln x)$$

for $1 < x < 2$.
In order to satisfy the second boundary condition $\phi(2) = 0$, we need

$$\phi(2) = B\sin(\lambda \ln 2) = 0,$$

and if $B = 0$ we get the trivial solution.
Therefore we have a nontrivial solution to the boundary value problem if and only if

$$\sin(\lambda \ln 2) = 0,$$

that is, if and only if $\lambda \ln 2 = n\pi$ for some integer n.

The eigenvalues and eigenfunctions for this boundary value problem are given by

$$\lambda_n^2 = \left(\frac{n\pi}{\ln 2}\right)^2 \qquad \text{and} \qquad \phi_n(x) = \sin\left(\frac{n\pi \ln x}{\ln 2}\right), \quad 1 < x < 2$$

for $n \geq 1$.

(b) From the differential equation, eigenfunctions corresponding to distinct eigenvalues will be orthogonal on the interval $[1, 2]$ with respect to the weight function $\sigma(x) = \dfrac{1}{x}$.

To show this directly, suppose that m and n are positive integers with $m \neq n$, then

$$\begin{aligned}
\int_1^2 \phi_m(x)\phi_n(x)\frac{1}{x}\,dx &= \int_1^2 \sin\left(\frac{m\pi \ln x}{\ln 2}\right)\sin\left(\frac{n\pi \ln x}{\ln 2}\right)\frac{1}{x}\,dx \\
&= \ln 2 \int_0^1 \sin(m\pi t)\sin(n\pi t)\,dt \qquad (t = \ln x/\ln 2) \\
&= 0
\end{aligned}$$

if $m \neq n$.

(c) Let $u(x)$ be a test function satisfying only the boundary conditions

$$u(1) = 0 \qquad \text{and} \qquad u(2) = 0,$$

the simplest such function is the quadratic

$$u(x) = (2 - x)(x - 1) = -x^2 + 3x - 2 \qquad \text{with} \qquad u'(x) = -2x + 3.$$

The Rayleigh quotient for this function is

$$R(u) = \frac{-p(x)u(x)u'(x)\Big|_1^2 + \displaystyle\int_1^2 \left[p(x)u'(x)^2 - q(x)u(x)^2\right]\,dx}{\displaystyle\int_1^2 u(x)^2\sigma(x)\,dx},$$

where $p(x) = x$, $q(x) = 0$, and $\sigma(x) = \dfrac{1}{x}$.

Computing $R(u)$, we have

$$\begin{aligned}
R(u) &= \frac{\displaystyle\int_1^2 xu'(x)^2\,dx}{\displaystyle\int_1^2 u(x)^2\sigma(x)\,dx} \\
&= \frac{\displaystyle\int_1^2 x(2x-3)^2\,dx}{\displaystyle\int_1^2 \left[(2-x)^2(x-1)^2/x\right]\,dx} \\
&= \frac{1/2}{-11/4 + 4\ln 2}
\end{aligned}$$

and since λ_1 is the minimum Rayleigh quotient over all such test functions, then

$$\lambda_1 \leq \frac{1/2}{-11/4 + 4\ln 2} \approx 23.$$

Question 4. Find the solution of the **exterior Dirichlet problem for a disk**, that is find a bounded solution to the problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \qquad a<r<\infty, \quad -\pi<\theta<\pi$$

$$u(r,\pi)=u(r,-\pi) \qquad a<r<\infty$$

$$\frac{\partial u}{\partial \theta}(r,\pi)=\frac{\partial u}{\partial \theta}(r,-\pi) \quad a<r<\infty$$

$$u(a,\theta)=f(\theta) \qquad -\pi<\theta<\pi.$$

SOLUTION: A solution to Laplace's equation in polar coordinates which satisfies the periodicity conditions is given by

$$u(r,\theta)=A_0+B_0\,\log r+\sum_{n=1}^{\infty}\left\{r^n\big(A_n\,\cos n\theta+B_n\,\sin n\theta\big)+\frac{1}{r^n}\big(C_n\,\cos n\theta+D_n\,\sin n\theta\big)\right\},$$

and in order to satisfy the boundedness condition we need $B_0=A_n=B_n=0$, for $n=1,2,3,\ldots$, so that

$$u(r,\theta)=A_0+\sum_{n=1}^{\infty}\frac{1}{r^n}\big(C_n\,\cos n\theta+D_n\,\sin n\theta\big).$$

Now, when $r=a$ we have

$$f(\theta)=u(a,\theta)=A_0+\sum_{n=1}^{\infty}\frac{1}{a^n}\big(C_n\,\cos n\theta+D_n\,\sin n\theta\big),$$

where

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\,d\phi,$$

$$C_n=\frac{a^n}{\pi}\int_{-\pi}^{\pi}f(\phi)\,\cos n\phi\,d\phi,$$

$$D_n=\frac{a^n}{\pi}\int_{-\pi}^{\pi}f(\phi)\,\sin n\phi\,d\phi$$

for $n=1,2,3\ldots.$

Therefore

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\,d\phi+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\int_{-\pi}^{\pi}f(\phi)\big\{\cos n\phi\,\cos n\theta+\sin n\phi\,\sin n\theta\big\}\,d\phi,$$

that is,

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\left\{1+2\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\theta-\phi)\right\}d\phi.$$

We can actually sum the series to get a much simpler expression for $u(r,\theta)$. Let $z=\frac{a}{r}\,e^{i(\theta-\phi)}$, then

$$z^n=\left(\frac{a}{r}\right)^n e^{in(\theta-\phi)}=\left(\frac{a}{r}\right)^n\left[\cos n(\theta-\phi)+i\,\sin n(\theta-\phi)\right],$$

and

$$1+2\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\theta-\phi)=\operatorname{Re}\left(1+2\sum_{n=1}^{\infty}z^n\right).$$

Since $|z| = \dfrac{a}{r} < 1$, then

$$1 + 2\sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos n(\theta - \phi) = \text{Re}\left(1 + \frac{2z}{1-z}\right) = \text{Re}\left(\frac{1+z}{1-z}\right) = \frac{r^2 - a^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}.$$

The solution to the exterior Dirichlet problem for the disk is therefore

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(r^2 - a^2)\, f(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\, d\phi,$$

for $a < r < \infty, \quad -\pi < \theta < \pi$.

Question 5. Find all functions ϕ for which $u(x,t) = \phi(x + ct)$ is a solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

where k and c are constants.

Solution: If $u(x,t) = \phi(x+ct)$ is a solution to the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t},$$

let $\xi = x + ct$, then from the chain rule we have

$$\frac{\partial u}{\partial x} = \frac{d\phi}{d\xi}\frac{\partial \xi}{\partial x} = \frac{d\phi}{d\xi},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d}{d\xi}\left(\frac{d\phi}{d\xi}\right)\frac{\partial \xi}{\partial x} = \frac{d^2\phi}{d\xi^2},$$

$$\frac{\partial u}{\partial t} = \frac{d\phi}{d\xi}\frac{\partial \xi}{\partial t} = c\,\frac{d\phi}{d\xi}.$$

Therefore, ϕ satisfies the ordinary differential equation

$$\frac{d^2\phi}{d\xi^2} - \frac{c}{k}\frac{d\phi}{d\xi} = 0,$$

and the solution is given by

$$\phi(\xi) = A + B\, e^{\frac{c}{k}\xi},$$

that is,

$$u(x,t) = A + B\, e^{\frac{c}{k}(x+ct)}$$

where A and B are arbitrary constants.

Question 6. Consider torsional oscillations of a homogeneous cylindrical shaft. If $\omega(x,t)$ is the angular displacement at time t of the cross section at x, then

$$\frac{\partial^2\omega}{\partial t^2} = a^2\frac{\partial^2\omega}{\partial x^2} \qquad 0 < x < L, \qquad t > 0.$$

Solve this problem if

$$\omega(x,0) = f(x) \qquad 0 < x < L$$

$$\frac{\partial\omega}{\partial t}(x,0) = 0 \qquad 0 < x < L,$$

and the ends of the shaft are fixed elastically:

$$\frac{\partial\omega}{\partial x}(0,t) - \alpha\,\omega(0,t) = 0 \quad t > 0$$

$$\frac{\partial\omega}{\partial x}(L,t) + \alpha\,\omega(L,t) = 0 \quad t > 0$$

with α a positive constant.

Solution: Since the partial differential equation is linear and homogeneous and the boundary conditions are linear and homogeneous, we can use separation of variables. Assuming a solution of the form

$$\omega(x,t) = \phi(x)\cdot G(t), \quad 0 \le x \le L, \quad t \ge 0$$

and separating variables, we have two ordinary differential equations:

$$\begin{aligned} \phi''(x) + \lambda\phi(x) = 0, \quad 0 \le x \le L, \qquad & G''(t) + \lambda a^2 G(t) = 0, \quad t > 0, \\ \phi'(0) - \alpha\phi(0) = 0 & \\ \phi'(L) + \alpha\phi(L) = 0 & \end{aligned}$$

We use the Rayleigh quotient to show that $\lambda > 0$ for all eigenvalues λ.

Let λ be an eigenvalue of the Sturm Liouville problem, and let $\phi(x)$ be the corresponding eigenfunction, then

$$-p(x)\phi(x)\phi'(x)\Big|_0^L = -\phi(L)\phi'(L) + \phi(0)\phi'(0) = \alpha(\phi(0)^2 + \phi(L)^2) > 0,$$

and since $q(x) = 0 \le 0$ for all $0 \le x \le L$, then

$$\lambda = \frac{\alpha(\phi(0)^2 + \phi(L)^2) + \displaystyle\int_0^L \phi'(x)^2\,dx}{\displaystyle\int_0^L \phi(x)^2\,dx} \ge 0$$

since $p(x) = \sigma(x) = 1$ for $0 \le x \le L$.

Note that if $\lambda = 0$, then

$$\alpha\big(\phi(0)^2 + \phi(L)^2\big) + \int_0^L \phi'(x)^2\,dx = 0$$

implies that

$$\alpha\big(\phi(0)^2 + \phi(L)^2\big) = 0 \qquad \text{and} \qquad \int_0^L \phi'(x)^2\,dx = 0.$$

Since $\alpha > 0$, this implies that $\phi(0) = 0$ and $\phi(L) = 0$; and since ϕ' is continuous on $[0,L]$, that $\phi'(x) = 0$ for $0 \le x \le L$. Therefore $\phi(x)$ is constant on $[0,L]$, so that $\phi(x) = \phi(0) = 0$ for $0 < x < L$, and $\lambda = 0$ is not an eigenvalue, and all of the eigenvalues λ of this Sturm-Liouville problem satisfy $\lambda > 0$.

If $\lambda > 0$, then $\lambda = \mu^2$, where $\mu \neq 0$, and the differential equation is $\phi'' + \mu^2\phi = 0$ with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x \qquad \text{and} \qquad \phi'(x) = -\mu A\sin\mu x + \mu B\cos\mu x$$

for $0 \leq x \leq L$.

From the first boundary condition

$$\phi'(0) - \alpha\phi(0) = \mu B - \alpha A = 0,$$

and $A = \dfrac{\mu B}{\alpha}$, and the solution is now

$$\phi(x) = B(\mu\cos\mu x + \alpha\sin\mu x).$$

From the second boundary condition

$$\phi'(L) + \alpha\phi(L) = B\left[-\mu^2\sin\mu L + \alpha\mu\cos\mu L + \alpha\mu\cos\mu L + \alpha^2\sin\mu L\right] = 0,$$

that is,

$$B\left[(\alpha^2 - \mu^2)\sin\mu L + 2\alpha\mu\cos\mu L\right] = 0,$$

and the boundary value problem has a nontrivial solution if and only if

$$\tan\mu L = \frac{2\alpha\mu}{\mu^2 - \alpha^2},$$

that is, if and only if

$$\tan\sqrt{\lambda}L = \frac{2\alpha\sqrt{\lambda}}{\lambda - \alpha^2}.$$

In order to determine the eigenvalues we sketch the graphs of the functions

$$f(\mu) = \tan\mu L \qquad \text{and} \qquad g(\mu) = \frac{2\alpha\mu}{\mu^2 - \alpha^2}$$

for $\mu > 0$.

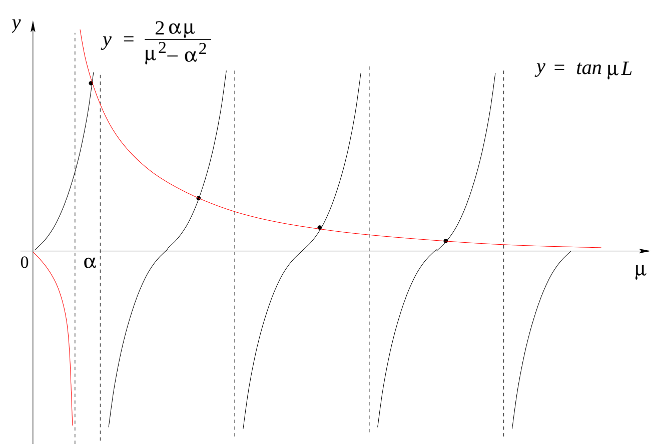
Note that for $\mu > 0$, we have

$$g(\mu) = \frac{2\alpha\mu}{\mu^2 - \alpha^2} = \alpha\left[\frac{1}{\mu + \alpha} + \frac{1}{\mu - \alpha}\right],$$

so that

$$g'(\mu) = -\alpha\left[\frac{1}{(\mu + \alpha)^2} + \frac{1}{(\mu - \alpha)^2}\right] < 0$$

and g is decreasing on the interval $(0, \alpha)$ and on the interval (α, ∞) and the line $\mu = \alpha$ is a vertical asymptote to the graph. The graphs of g and f are shown below.



From the figure it is clear that there are an infinite number of distinct solutions μ_n to the equation

$$\tan \mu L = \frac{2\alpha\mu}{\mu^2 - \alpha^2},$$

and the eigenvalues are $\lambda_n = \mu_n^2$, for $n \geq 1$, while the corresponding eigenfunctions are

$$\phi_n(x) = \mu_n \cos \mu_n x + \alpha \sin \mu_n x, \quad 0 \leq x \leq L$$

for $n \geq 1$.

The corresponding solutions to the time equation are

$$G_n(t) = a_n \cos \mu_n a t + b_n \sin \mu_n a t, \quad t \geq 0$$

and from the superposition principle, the function

$$\omega(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \cdot G_n(t) = \sum_{n=1}^{\infty} \left(\mu_n \cos \mu_n x + \alpha \sin \mu_n x\right)\left(a_n \cos \mu_n a t + b_n \sin \mu_n a t\right)$$

satisfies the partial differential equation and the boundary conditions.

Since the spatial problem is a regular Sturm-Liouville problem, then the eigenfunctions are orthogonal on the interval $[0, L]$, and we use this fact to satisfy the initial conditions

$$\omega(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad \text{and} \qquad \frac{\partial \omega}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \mu_n \phi_n(x) = 0,$$

and the generalized Fourier coefficients are given by

$$a_n = \frac{\displaystyle\int_0^L f(x)\phi_n(x)\,dx}{\displaystyle\int_0^L \phi_n(x)^2\,dx} \qquad \text{and} \qquad b_n = 0$$

for $n \geq 1$.

Therefore the solution is

$$\omega(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) \cos \mu_n a t, \quad 0 \leq x \leq L,\ t \geq 0$$

where

$$a_n = \frac{\displaystyle\int_0^L f(x)\phi_n(x)\,dx}{\displaystyle\int_0^L \phi_n(x)^2\,dx}$$

for $n \geq 1$.

Question 7. Use D'Alembert's solution of the wave equation to solve the initial value - boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad -\infty < x < \infty, \qquad t > 0$$
$$u(x,0) = f(x) \qquad -\infty < x < \infty$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad -\infty < x < \infty$$

with $f(x) = 0$ and $g(x) = \dfrac{x}{1+x^2}$.

Solution: The initial value - boundary value problem for the displacement of an infinite vibrating string is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad -\infty < x < \infty, \qquad t > 0$$
$$u(x,0) = f(x) \qquad -\infty < x < \infty$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad -\infty < x < \infty$$

and the general solution, that is, d'Alembert's solution to the wave equation, is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

for $-\infty < x < \infty, \quad t > 0$, and since $f(x) = 0$ for $-\infty < x < \infty$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$
$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{s}{1+s^2}\,ds.$$

The solution to the wave problem is therefore

$$u(x,t) = \frac{1}{4c}\left[\ln(1+(x+ct)^2) - \ln(1+(x-ct)^2)\right] = \frac{1}{4c}\ln\left[\frac{1+(x+ct)^2}{1+(x-ct)^2}\right],$$

for $-\infty < x < \infty, \quad t > 0$.

Question 8. Obtain the expansion

$$e^{ax} = \frac{\sinh \pi a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)$$

valid for all real numbers $a \neq 0$, and all $-\pi < x < \pi$.

Solution: Let

$$f(x) = e^{ax}, \quad -\pi < x < \pi, \quad f(x+2\pi) = f(x), \quad a \neq 0,$$

then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{\sinh \pi a}{\pi a}.$$

For $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx \, dx = 2a \sinh \pi a \frac{(-1)^n}{\pi(n^2 + a^2)},$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin nx \, dx = -2n \sinh \pi a \frac{(-1)^n}{\pi(n^2 + a^2)},$$

so the Fourier series is for $f(x)$ on the interval $-\pi < x < \pi$ is given by

$$e^{ax} = \frac{2}{\pi} \sinh \pi a \left\{ \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} (a \cos nx - n \sin nx) \right\}.$$

Therefore,

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} (a \cos nx - n \sin nx) + \frac{\sinh \pi a}{\pi} \frac{1}{a} + \frac{\sinh \pi a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} (a \cos nx - n \sin nx),$$

and replacing n by $-n$ in the first sum, we have

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{-1} \frac{(-1)^n}{n^2 + a^2} (a \cos nx - n \sin nx) + \frac{\sinh \pi a}{\pi} \frac{1}{a} + \frac{\sinh \pi a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} (a \cos nx - n \sin nx),$$

that is,

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} (a \cos nx - n \sin nx)$$

for $-\pi < x < \pi$.

Question 9. Consider the regular Sturm-Liouville problem

$$\left(x^2 X'\right)' + \lambda X = 0 \quad 1 < x < e$$
$$X(1) = 0$$
$$X(e) = 0$$

(a) Show that the substitution

$$X = \frac{Y}{\sqrt{x}}$$

transforms this problem into the following problem

$$x\left(xY'\right)' + \mu Y = 0 \quad 1 < x < e$$
$$Y(1) = 0$$
$$Y(e) = 0$$

where $\mu = \lambda - \dfrac{1}{4}$.

(b) Let $x = e^t$ and $\widehat{Y}(t) = Y(e^t)$, show that this transforms the problem in part (a) into the problem

$$\frac{d^2\widehat{Y}}{dt^2} + \mu \widehat{Y} = 0 \quad 0 < t < 1$$
$$\widehat{Y}(0) = 0$$
$$\widehat{Y}(1) = 0.$$

(c) Find the eigenvalues and eigenvectors for the problem in part (b), and from these, the eigenvalues and eigenfunctions in part (a), and finally obtain the eigenvalues and eigenfunctions for the original Sturm-Liouville problem.

Solution:

(a) Let

$$X = \frac{Y}{x^{\frac{1}{2}}},$$

then

$$X' = \frac{Y'}{x^{\frac{1}{2}}} - \frac{1}{2}\frac{Y}{x^{\frac{3}{2}}},$$

so that

$$\begin{aligned} x^2 X' &= x^{\frac{3}{2}} Y' - \tfrac{1}{2} x^{\frac{1}{2}} Y \\ &= x^{\frac{1}{2}} x Y' - \tfrac{1}{2} x^{\frac{1}{2}} Y. \end{aligned}$$

Differentiating again and simplifying, we have

$$\left(x^2 X'\right)' = x^{-\frac{1}{2}} \left\{ x\,(x\,Y')' - \tfrac{1}{4} Y \right\}.$$

Therefore,

$$\left(x^2 X'\right)' + \lambda X = x^{-\frac{1}{2}} \left\{ x\left(x Y'\right)' + \mu Y \right\}$$

where $\mu = \lambda - \frac{1}{4}$, and the original problem is transformed into the problem:

$$x\left(x Y'\right)' + \mu Y = 0, \quad 1 < x < e$$
$$Y(1) = 0$$
$$Y(e) = 0$$

where $\mu = \lambda - \frac{1}{4}$.

(b) Now let $x = e^t$ and $\hat{Y}(t) = Y(e^t)$, then

$$\frac{dY}{dx} = \frac{d\hat{Y}}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{d\hat{Y}}{dt},$$

that is,

$$x\frac{dY}{dx} = \frac{d\hat{Y}}{dt}.$$

Also,

$$\frac{d}{dx}\left(x\frac{dY}{dx}\right) = \frac{d}{dt}\left(\frac{d\hat{Y}}{dt}\right)\frac{dt}{dx} = \frac{1}{x}\frac{d^2\hat{Y}}{dt^2} = e^{-t}\frac{d^2\hat{Y}}{dt^2},$$

and therefore

$$\frac{d}{dx}\left(x\frac{dY}{dx}\right) + \frac{\mu}{x}Y = 0 \quad \text{for} \quad 1 < x < e$$

if and only if

$$e^{-t}\frac{d^2\hat{Y}}{dt^2} + e^{-t}\mu\hat{Y} = 0 \quad \text{for} \quad 0 < t < 1$$

So we have the equivalent regular Sturm-Liouville problem

$$\frac{d^2\hat{Y}}{dt^2} + \mu\hat{Y} = 0, \quad 0 < t < 1$$
$$\hat{Y}(0) = 0$$
$$\hat{Y}(1) = 0$$

with eigenvalues $\mu_n = n^2\pi^2$, and eigenfunctions

$$\hat{Y}_n(t) = \sin n\pi t, \quad 0 < t < 1$$

for $n \geq 1$.

(c) The eigenvalues for the original problem are therefore

$$\lambda_n = n^2\pi^2 + \tfrac{1}{4},$$

and the corresponding eigenfunctions are

$$X_n(x) = \frac{1}{\sqrt{x}}\sin(n\pi\log x), \quad 1 < x < e$$

for $n \geq 1$.

Question 10. Show that if $|a| < 1$, then

(a) $\sum\limits_{n=1}^{\infty} a^n \cos nx = \dfrac{a\cos x - a^2}{1 - 2a\cos x + a^2}$ for $-\pi < x < \pi$,

(b) $\sum\limits_{n=1}^{\infty} a^n \sin nx = \dfrac{a\sin x}{1 - 2a\cos x + a^2}$ for $-\pi < x < \pi$,

Solution: Let $z = a\, e^{ix}$, then $|z| = |a| < 1$, so the geometric series

$$\sum_{n=0}^{\infty} a^n e^{inx}$$

converges and

$$\sum_{n=0}^{\infty} a^n e^{inx} = \frac{1}{1 - ae^{ix}} = \frac{1}{(1 - a\cos x) - ia\sin x} = \frac{1 - a\cos x}{1 - 2a\cos x + a^2} + i\frac{a\sin x}{1 - 2a\cos x + a^2}.$$

Since

$$\sum_{n=0}^{\infty} a^n e^{inx} = 1 + \sum_{n=1}^{\infty} a^n \cos nx + i\sum_{n=1}^{\infty} a^n \sin nx,$$

then

$$1 + \sum_{n=1}^{\infty} a^n \cos nx + i\sum_{n=1}^{\infty} a^n \sin nx = \frac{1 - a\cos x}{1 - 2a\cos x + a^2} + i\frac{a\sin x}{1 - 2a\cos x + a^2},$$

and equating real and imaginary parts, we have

$$\sum_{n=1}^{\infty} a^n \cos nx = \frac{1 - a\cos x}{1 - 2a\cos x + a^2} - 1 = \frac{a\cos x - a^2}{1 - 2a\cos x + a^2},$$

and

$$\sum_{n=1}^{\infty} a^n \sin nx = \frac{a\sin x}{1 - 2a\cos x + a^2}$$

for $-\pi < x < \pi$.

Question 11. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 - \cos x & \text{if } \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: The Fourier integral representation of $f(x)$ is given by

$$f(x) \sim \int_0^{\infty} \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)\, d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt \qquad \text{and} \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt.$$

Since $f(x)$ is an even function, then $B(\omega) = 0$ for all ω.

Also, since $f(x)$ is even and $f(x) = 0$ for $|x| \geq \frac{\pi}{2}$, then for all $\omega \neq 0$ and $\omega \neq \pm 1$, we have

$$
\begin{aligned}
A(\omega) &= \frac{2}{\pi} \int_0^{\pi/2} (1 - \cos t) \cos \omega t \, dt \\
&= \frac{2}{\pi} \int_0^{\pi/2} \cos \omega t \, dt - \frac{2}{\pi} \int_0^{\pi/2} \cos t \cos \omega t \, dt \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi} \int_0^{\pi/2} \left[\cos(1-\omega)t + \cos(1+\omega)t\right] \, dt \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi} \frac{\sin(1-\omega)t}{1-\omega} \bigg|_0^{\pi/2} - \frac{1}{\pi} \frac{\sin(1+\omega)t}{1+\omega} \bigg|_0^{\pi/2} \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{1}{\pi} \frac{\sin((1-\omega)\pi/2)}{1-\omega} - \frac{1}{\pi} \frac{\sin((1+\omega)\pi/2)}{1+\omega} \\
&= \frac{2}{\pi} \frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{\pi} \left[\frac{1}{1-\omega} + \frac{1}{1+\omega}\right] \\
&= \frac{2}{\pi} \left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right],
\end{aligned}
$$

so that

$$
A(\omega) = \frac{2}{\pi} \left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right]
$$

for $\omega \neq 0, \pm 1$.

If $\omega = 0$, then

$$
A(0) = \frac{2}{\pi} \int_0^{\pi/2} (1 - \cos t) \, dt = \frac{2}{\pi} \left[\frac{\pi}{2} - \sin(\pi/2)\right] = 1 - \frac{2}{\pi}.
$$

If $\omega = \pm 1$, then

$$
A(\pm 1) = \frac{2}{\pi} \frac{\sin(\pm\pi/2)}{\pm 1} - \frac{2}{\pi} \int_0^{\pi/2} \cos^2 t \, dt = \frac{2}{\pi} - \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{1 + \cos 2t}{2}\right) dt = \frac{2}{\pi} - \frac{1}{2}.
$$

Note that $A(w)$ is continuous for all ω.

From Dirichlet's convergence theorem, the integral

$$
\frac{2}{\pi} \int_0^{\infty} \left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right] \cos \omega x \, d\omega
$$

converges to $1 - \cos x$ for all $|x| < \frac{\pi}{2}$, converges to 0 for all $|x| > \frac{\pi}{2}$, and converges to $\frac{1}{2}$ for $x = \pm\frac{\pi}{2}$.

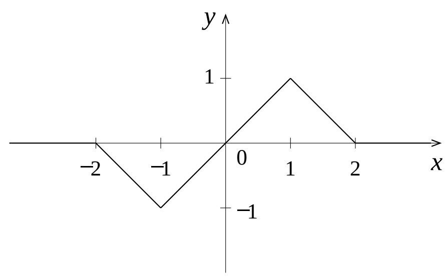
Thus, if we redefine $f(\pm\pi/2) = \frac{1}{2}$, then the Fourier integral representation of $f(x)$ i s given by

$$
\frac{2}{\pi} \int_0^{\infty} \left[\frac{\sin(\omega\pi/2)}{\omega} - \frac{\cos(\omega\pi/2)}{1-\omega^2}\right] \cos \omega x \, d\omega = f(x) = \begin{cases} 1 - \cos x & \text{for} \quad |x| < \frac{\pi}{2} \\ 0 & \text{for} \quad |x| > \frac{\pi}{2} \\ \frac{1}{2} & \text{for} \quad x = \pm\frac{\pi}{2}. \end{cases}
$$

Question 12. Find the Fourier integral representation of the function

$$f(x) = \begin{cases} x & \text{if } \quad -1 < x < 1, \\ 2 - x & \text{if } \quad 1 < x < 2, \\ -2 - x & \text{if } \quad -2 < x < -1, \\ 0 & \text{otherwise.} \end{cases}$$

SOLUTION: The graph of $f(x)$ is shown below and it is easy to see that the function $f(x)$ is an odd function.



Therefore, $A(\omega) = 0$ for all ω, and

$$B(\omega) = \frac{2}{\pi} \int_0^2 f(t) \sin \omega t \, dt = \frac{2}{\pi} \int_0^1 t \sin \omega t \, dt + \frac{2}{\pi} \int_1^2 (2 - t) \sin \omega t \, dt.$$

Therefore, integrating by parts, we have

$$\begin{aligned} B(\omega) &= \frac{2}{\pi} \left[\frac{-t}{\omega} \cos \omega t \Big|_0^1 + \int_0^1 \frac{\cos \omega t}{\omega} \right] + \frac{2}{\pi} \left[\frac{-2 + t}{\omega} \cos \omega t \Big|_1^2 - \int_1^2 \frac{\cos \omega t}{\omega} \, dt \right] \\ &= \frac{2}{\pi} \left[-\frac{\cos \omega}{\omega} + \frac{\sin \omega t}{\omega^2} \Big|_0^1 \right] + \frac{2}{\pi} \left[\frac{\cos \omega}{\omega} - \frac{\sin \omega t}{\omega^2} \Big|_1^2 \right] \\ &= \frac{2}{\pi} \left[\frac{2 \sin \omega}{\omega^2} - \frac{\sin 2\omega}{\omega^2} \right] \\ &= \frac{2}{\pi} \left(\frac{2 \sin \omega - \sin 2\omega}{\omega^2} \right), \end{aligned}$$

that is,

$$B(\omega) = \frac{2}{\pi} \left(\frac{2 \sin \omega - \sin 2\omega}{\omega^2} \right)$$

for all $\omega \neq 0$.

If $\omega = 0$, then

$$B(0) = \frac{2}{\pi} \int_0^2 f(t) \sin(0 \cdot t) \, dt = 0.$$

Since $f(x)$ is continuous everywhere, from Dirichlet's convergence theroem, the Fourier sine integral converges to $f(x)$ for all x, and therefore

$$\frac{2}{\pi} \int_0^\infty \left(\frac{2 \sin \omega - \sin 2\omega}{\omega^2} \right) \sin \omega x \, d\omega = f(x)$$

for all $x \in \mathbb{R}$.

Question 13. Let

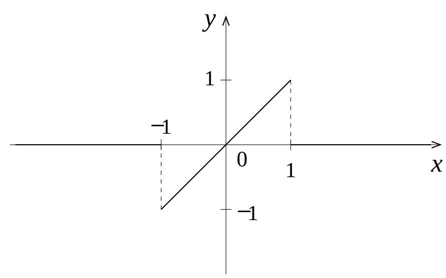
$$f(x) = \begin{cases} x & \text{if } \quad |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Plot the function $f(x)$ and find its Fourier transform.

(b) If $\widehat{f}$ is real valued, plot it; otherwise plot $|\widehat{f}|$.

SOLUTION:

(a) The graph of the function $f(x)$ is plotted below.



The Fourier transform of $f(x)$ is computed as

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = \frac{1}{2\pi}\int_{-1}^{1} te^{-i\omega t}\,dt \\
&= \frac{1}{2\pi}\left[-\frac{t}{i\omega}e^{-i\omega t}\Big|_{-1}^{1} + \frac{1}{i\omega}\int_{-1}^{1} e^{-i\omega t}\,dt\right] \\
&= \frac{1}{2\pi}\left[-\frac{1}{i\omega}\left(e^{-i\omega} + e^{i\omega}\right) - \frac{1}{(i\omega)^2}e^{-i\omega t}\Big|_{-1}^{1}\right] \\
&= \frac{2i}{2\pi}\left[\left(\frac{e^{i\omega} + e^{-i\omega}}{2\omega}\right) - \left(\frac{e^{i\omega} - e^{-i\omega}}{2i\omega^2}\right)\right] \\
&= \frac{i}{\pi}\left(\frac{\omega\cos\omega - \sin\omega}{w^2}\right),
\end{aligned}$$

so that

$$\widehat{f}(\omega) = \frac{i}{\pi}\left(\frac{\omega\cos\omega - \sin\omega}{w^2}\right)$$

for all $\omega \neq 0$.

If $\omega = 0$, then

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-1}^{1} t\,dt = \frac{1}{2\pi}\frac{t^2}{2}\Big|_{-1}^{1} = 0,$$

and from L'Hospital's rule, we see that $\lim_{\omega\to 0}\widehat{f}(\omega) = 0$ also, so that $\widehat{f}(\omega)$ is continuous at each ω.

(b) Since

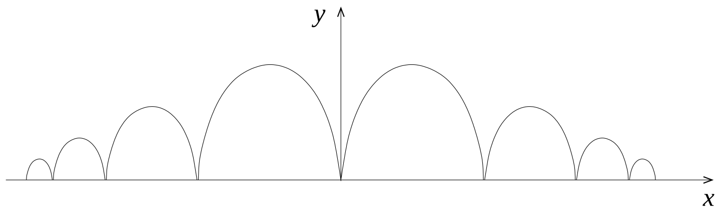
$$\widehat{f}(\omega) = \frac{i}{\pi}\left(\frac{\omega\cos\omega - \sin\omega}{w^2}\right),$$

then

$$\left|\widehat{f}(\omega)\right| = \frac{1}{\pi}\left|\frac{\sin\omega - \omega\cos\omega}{\omega^2}\right|$$

for all ω.

Note that the zeros of the function $g(\omega) = \sin\omega - \omega\cos\omega$ are precisely the roots of the equation $\tan\omega = \omega$, so the graph of $\left|\widehat{f}(\omega)\right|$ looks something like the figure below.



Question 14. Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1 - x & \text{if} \quad 0 < x < 1, \\ 0 & \text{if} \quad x \geq 1. \end{cases}$$

and write $f(x)$ as an inverse cosine transform. Use a known Fourier transform and the fact that if $f(x),\ x \geq 0$, is the restriction of an *even* function f_e, then

$$\mathcal{F}_c(f)(\omega) = 2\mathcal{F}(f_e)(\omega)$$

for all $\omega \geq 0$.

SOLUTION: The Fourier cosine transform of the function f is given by

$$\widehat{f_c}(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt = \frac{2}{\pi}\int_0^1 (1-t)\cos\omega t\,dt,$$

and this is the same as the Fourier transform of the *even* extension f_e of f to the whole real line $\mathbb{R}$.

In this case however, we can evaluate the last integral directly by integration by parts:

$$\begin{aligned}\int_0^1 (1-t)\cos\omega t\,dt &= \int_0^1 \cos\omega t\,dt - \int_0^1 t\cos\omega t\,dt \\ &= \frac{\sin\omega t}{\omega}\bigg|_0^1 - \left[t\cdot\frac{\sin\omega t}{\omega}\bigg|_0^1 - \frac{1}{\omega}\int_0^1 \sin\omega t\,dt\right] \\ &= \frac{\sin\omega}{\omega} - \frac{\sin\omega}{\omega} + \frac{1}{\omega}\left[-\frac{1}{\omega}\cos\omega t\bigg|_0^1\right] \\ &= \frac{1-\cos\omega}{\omega^2},\end{aligned}$$

and therefore

$$\widehat{f_c}(\omega) = \frac{2}{\pi}\cdot\frac{1-\cos\omega}{\omega^2}$$

for $\omega > 0$.

Knowing that f_c is absolutely integrable implies that $\widehat{f_c}$ is continuous at $\omega = 0$, and we have

$$\widehat{f_c}(0) = \lim_{\omega\to 0+}\frac{2}{\pi}\cdot\frac{1-\cos\omega}{\omega^2} = \frac{2}{\pi}\cdot\lim_{\omega\to 0+}\frac{\sin\omega}{2\omega} = \frac{1}{\pi}$$

by L'Hospital's rule.

Therefore, we have

$$\widehat{f}_c(\omega) = \begin{cases} \dfrac{2}{\pi} \cdot \dfrac{1-\cos\omega}{\omega^2} & \text{for} \quad \omega > 0 \\ \dfrac{1}{\pi} & \text{for} \quad \omega = 0. \end{cases}$$

Since f_e is continuous for all $x \in \mathbb{R}$, from Dirichlet's theorem the inverse Fourier cosine transform of $\widehat{f}_c$ is given by

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\omega}{\omega^2} \cdot \cos\omega x \, d\omega = \begin{cases} 1-x & \text{for} \quad 0 \le x < 1 \\ 0 & \text{for} \quad x \ge 1. \end{cases}$$

Question 15. Find the Fourier sine transform of

$$f(x) = \frac{x}{1+x^2}, \quad x > 0,$$

and write $f(x)$ as an inverse sine transform. Use a known Fourier transform and the fact that if $f(x)$, $x \ge 0$, is the restriction of an *odd* function f_o, then

$$\mathcal{F}_s(f)(\omega) = -2i\mathcal{F}(f_o)(\omega)$$

for all $\omega \ge 0$.

Solution: We can find the Fourier sine transform of the given function using the suggested method, or we can find it directly. To do this, we consider the function

$$g(x) = e^{-x}, \quad x > 0$$

with Fourier sine transform given by

$$\widehat{g}_s(\omega) = \frac{2}{\pi}\int_0^\infty e^{-t}\sin\omega t \, dt$$

and we can evaluate this integral by integrating by parts:

$$\begin{aligned}
\int_0^\infty e^{-t}\sin\omega t \, dt &= -\frac{e^{-t}}{\omega}\bigg|_0^\infty - \frac{1}{\omega}\int_0^\infty e^{-t}\cos\omega t \, dt \\
&= \frac{1}{\omega} - \frac{1}{\omega}\left[e^{-t} \cdot \frac{\sin\omega t}{\omega}\bigg|_0^\infty + \frac{1}{\omega}\int_0^\infty e^{-t}\sin\omega t \, dt\right] \\
&= \frac{1}{\omega} - \frac{1}{\omega^2}\int_0^\infty e^{-t}\sin\omega t \, dt
\end{aligned}$$

so that

$$\left(1 + \frac{1}{\omega^2}\right)\int_0^\infty e^{-t}\sin\omega t \, dt = \frac{1}{\omega}.$$

Therefore,

$$\int_0^\infty e^{-t}\sin\omega t \, dt = \frac{\omega}{1+\omega^2}$$

for $\omega \ge 0$, so that

$$\widehat{g}_s(\omega) = \frac{2}{\pi} \cdot \frac{\omega}{1+\omega^2}$$

for $\omega \ge 0$.

Taking the inverse Fourier sine transform of this, we have

$$g(x) = \int_0^\infty \widehat{g}_s(\omega) \sin \omega x \, d\omega = \frac{2}{\pi} \int_0^\infty \frac{\omega}{1+\omega^2} \sin \omega x \, d\omega,$$

that is,

$$e^{-\omega} = g(\omega) = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} \sin \omega x \, dx,$$

and

$$\widehat{f}_s(\omega) = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} \sin \omega x \, dx = g(\omega) = e^{-\omega}$$

for $\omega \geq 0$.

From the above, we can write $f(x)$ as an inverse Fourier sine transform:

$$f(x) = \frac{x}{1+x^2} = \int_0^\infty e^{-\omega} \sin \omega x \, d\omega$$

for $x > 0$.

Question 16. Use the Fourier transform to solve the heat flow problem in an infinite rod

$$\frac{\partial u}{\partial t} = 10 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \begin{cases} 2 & \text{for} \quad -\pi \leq x \leq \pi \\ 0 & \text{otherwise,} \end{cases}$$

and express the solution as the difference of two error functions.

Solution: The solution is the convolution

$$u(x,t) = f * G\,(x,t)$$

where

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$$

is the heat kernel, or Gaussian kernel, so that

$$u(x,t) = \frac{1}{\sqrt{40\pi t}} \int_{-\pi}^{\pi} 2e^{-\frac{(x-s)^2}{40t}} \, ds$$

for $-\infty < x < \infty,\ t > 0$.

Making the substitution

$$z = \frac{x-s}{\sqrt{40t}} \qquad \text{and} \qquad dz = -\frac{1}{\sqrt{40t}} ds,$$

when $s = -\pi$, then $z = \dfrac{x+\pi}{\sqrt{40t}}$, and when $s = \pi$, then $z = \dfrac{x-\pi}{\sqrt{40t}}$, so that

$$\begin{aligned} u(x,t) &= \frac{2}{\sqrt{40\pi t}} (-\sqrt{40t}) \int_{\frac{x+\pi}{\sqrt{40t}}}^{\frac{x-\pi}{\sqrt{40t}}} e^{-z^2} \, dz \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^{\frac{x+\pi}{\sqrt{40t}}} e^{-t^2} \, dt - \int_0^{\frac{x-\pi}{\sqrt{40t}}} e^{-t^2} \, dt \right) \\ &= \operatorname{erf}\left(\frac{x+\pi}{\sqrt{40t}} \right) - \operatorname{erf}\left(\frac{x-\pi}{\sqrt{40t}} \right). \end{aligned}$$

Therefore

$$u(x,t) = \operatorname{erf}\left(\frac{x+\pi}{\sqrt{40t}}\right) - \operatorname{erf}\left(\frac{x-\pi}{\sqrt{40t}}\right)$$

for $-\infty < x < \infty,\ t > 0$, where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$$

is the error function.

Question 17. Find the solution to the initial value problem

$$\frac{\partial u}{\partial t} + 5\frac{\partial u}{\partial x} = e^{3t},\ -\infty < x < \infty,\ t \geq 0$$

$$u(x,0) = e^{-x^2}, \quad -\infty < x < \infty$$

using the method of characteristics.

Solution: Let

$$\frac{dx}{dt} = 5,$$

then along the characteristic curve $x(t) = 5t + a$, the partial differential equation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = e^{3t},$$

so that

$$u(x(t),t) = \frac{1}{3}e^{3t} + K$$

where K is a constant, and $K = u(x(0),0) - \frac{1}{3}$, so that

$$u(x(t),t) = \frac{1}{3}e^{3t} + u(x(0),0) - \frac{1}{3} = \frac{1}{3}e^{3t} + u(a,0) - \frac{1}{3} = \frac{1}{3}e^{3t} + e^{-a^2} - \frac{1}{3}.$$

Given the point (x,t), let $x = 5t + a$ be the unique characteristic curve passing through this point, then

$$u(x,t) = \frac{1}{3}e^{3t} + e^{-a^2} - \frac{1}{3} = \frac{1}{3}e^{3t} + e^{-(x-5t)^2} - \frac{1}{3}.$$