Math 300
Solutions to Midterm Examination II

Question 2. Consider the regular Sturm-Liouville problem

$$\phi'' + \lambda^2 \phi = 0 \qquad 0 \le x \le \pi$$
$$\phi'(0) = 0$$
$$\phi(\pi) = 0$$

(a) Find the eigenvalues λ_n^2 and the corresponding eigenfunctions ϕ_n for this problem.

(b) Show directly, by integration, that eigenfunctions corresponding to distinct eigenvalues are orthogonal.

(c) Given the function $f(x) = \dfrac{\pi^2 - x^2}{2}, \quad 0 < x < \pi$, find the eigenfunction expansion for f.

(d) Show that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots$$

SOLUTION:

(a) We consider the cases where $\lambda = 0$ and $\lambda \neq 0$ separately.

If $\lambda = 0$, the general solution to the equation

$$\phi'' + \lambda^2 \phi = 0$$

in this case is

$$\phi(x) = c_1 x + c_2,$$

and differentiating,

$$\phi'(x) = c_1.$$

The condition $\phi'(0) = 0$ implies that $c_1 = 0$, while the condition $\phi(\pi) = 0$ implies that $c_2 = 0$, so there are no non-trivial solutions in this case.

If $\lambda \neq 0$, the general solution to the equation

$$\phi'' + \lambda^2 \phi = 0$$

in this case is

$$\phi(x) = c_1 \cos \lambda x + c_2 \sin \lambda x,$$

and differentiating, we get

$$\phi'(x) = -c_1 \lambda \sin \lambda x + c_2 \lambda \cos \lambda x.$$

The condition $\phi'(0) = 0$ implies that $c_2 \lambda = 0$, and so $c_2 = 0$. The solution is then

$$\phi(x) = c_1 \cos \lambda x,$$

and the condition $\phi(\pi) = 0$ implies that $\cos \lambda \pi = 0$, and therefore the eigenvalues are

$$\lambda^2 = \lambda_n^2 = \left(\tfrac{2n-1}{2}\right)^2,$$

for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are

$$\phi_n(x) = \cos \tfrac{(2n-1)}{2} x,$$

for $n = 1, 2, 3, \ldots$.

(b) For $n = 1, 2, 3, \ldots$, let $\lambda_n = \frac{2n-1}{2}$, if $m \neq n$, we have

$$\begin{aligned}
\int_0^\pi \phi_m(x)\phi_n(x)\,dx &= \int_0^\pi \cos\lambda_m x\,\cos\lambda_n x\,dx \\
&= \frac{1}{2}\int_0^\pi \{\cos(\lambda_m+\lambda_n)x + \cos(\lambda_m-\lambda_n)x\}\,dx \\
&= \frac{1}{2(\lambda_m+\lambda_n)}\sin(\lambda_m+\lambda_n)x\Big|_0^\pi + \frac{1}{2(\lambda_m-\lambda_n)}\sin(\lambda_m-\lambda_n)x\Big|_0^\pi \\
&= \frac{1}{2(\lambda_m+\lambda_n)}\sin(\lambda_m+\lambda_n)\pi + \frac{1}{2(\lambda_m-\lambda_n)}\sin(\lambda_m-\lambda_n)\pi \\
&= 0
\end{aligned}$$

since $(\lambda_m + \lambda_n)\pi = (m + n - 1)\pi$ and $(\lambda_m - \lambda_n)\pi = (m - n)\pi$.

(c) Writing

$$f(x) = \frac{\pi^2 - x^2}{2} \sim \sum_{n=1}^{\infty} c_n\phi_n(x),$$

the coefficients c_n in the eigenfunction expansion are found using the orthogonality of the eigenfunctions on $[0, \pi]$.

$$\begin{aligned}
c_n &= \frac{2}{\pi}\int_0^\pi \left(\frac{\pi^2 - x^2}{2}\right)\cos\lambda_n x\,dx \\
&= \frac{2}{\pi}\,\frac{\sin\lambda_n\pi}{\lambda_n^3} = \frac{16}{\pi(2n-1)^3}\sin\frac{(2n-1)}{2}\pi \\
&= \frac{16(-1)^{n+1}}{\pi(2n-1)^3}.
\end{aligned}$$

Therefore, the eigenfunction expansion of f is given by

$$\frac{\pi^2 - x^2}{2} \sim \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\cos\frac{(2n-1)}{2}x$$

for $0 \le x \le \pi$.

(d) For this particular problem, the eigenfunction expansion is actually the Fourier cosine series for f. Since the function f is piecewise smooth on the interval $[0, \pi]$ and since the even extension of f to $[-\pi, \pi]$ is continuous at $x = 0$, then by Dirichlet's theorem the series converges to

$$f(0) = \frac{\pi^2}{2}$$

when $x = 0$, and therefore

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \cdots$$

Question 3. Solve the following boundary value problem for the steady-state temperature $u(x,y)$ in a thin plate in the shape of a semi-infinite strip when heat transfer to the surroundings at temperature zero takes place at the faces of the plate:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - b\,u = 0, \qquad 0 < x < \infty, \quad 0 < y < 1$$

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad 0 < y < 1$$

$$u(x,0) = 0, \qquad 0 < x < \infty$$

$$u(x,1) = f(x), \quad 0 < x < \infty$$

where b is a positive constant and

$$f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a. \end{cases}$$

Solution: We try separation of variables. If we assume that

$$u(x,y) = X(x)\,Y(y),$$

then the partial differential equation becomes

$$X''Y + XY'' - bXY = 0,$$

that is,

$$\frac{X''}{X} = -\frac{Y''}{Y} + b = p \qquad \text{(constant)}$$

and we obtain the two ordinary differential equations

$$X'' - p\,X = 0 \quad 0 < x < \infty \qquad Y'' + (p-b)\,Y = 0, \quad 0 < y < 1$$

$$X'(0) = 0 \qquad\qquad Y(0) = 0$$

$$|X(x)| \quad \text{bounded as} \quad x \to \infty$$

We solve the X-equation first, and consider three cases:

(i) If $p = 0$, then the general solution to the equation $X'' = 0$ is

$$X(x) = c_1\,x + c_2$$

and the condition $X'(0) = 0$ implies that $c_1 = 0$, the solution is therefore $X(x) = 1$.

(ii) If $p > 0$, say $p = \mu^2$, then the general solution to the equation $X'' - \mu^2\,X = 0$ is

$$X(x) = c_1\,\cosh \mu x + c_2\,\sinh \mu x$$

and the condition $X'(0) = 0$ implies $c_2 = 0$, while the condition $|X(x)|$ bounded as $x \to \infty$ implies that $c_1 = 0$. There are no non-trivial solutions in this case.

(iii) If $p < 0$, say $p = -\lambda^2$, then the general solution to the equation $X'' + \lambda^2\,X = 0$ is

$$X(x) = c_1\,\cos \lambda x + c_2\,\sin \lambda x$$

the condition $X'(0) = 0$ implies that $c_2 = 0$, and the solution is $X(x) = c_1\,\cos \lambda x$, which is bounded as $x \to \infty$.

Therefore, for any $\lambda \geq 0$ the function

$$X_\lambda(x) = \cos \lambda x$$

satisfies the differential equation, the boundary condition, and the boundedness condition. The corresponding equation for Y is given by

$$\begin{aligned} Y'' - (\lambda^2 + b)\, Y &= 0 \\ Y(0) &= 0 \end{aligned}$$

and has general solution

$$Y(y) = c_1 \sinh\left((1-y)\sqrt{\lambda^2 + b}\right) + c_2 \sinh\left(y\sqrt{\lambda^2 + b}\right).$$

Now, the condition $Y(0) = 0$ implies that $c_1 = 0$, and the solutions are

$$Y_\lambda(y) = \sinh\left(y\sqrt{\lambda^2 + b}\right).$$

Using the superposition principle, we write

$$u(x, y) = \int_0^\infty A(\lambda) \cos \lambda x \sinh\left(y\sqrt{\lambda^2 + b}\right)\, d\lambda$$

and $u(x, 1) = f(x)$ implies that

$$\begin{aligned} A(\lambda) \sinh \sqrt{\lambda^2 + b} &= \frac{2}{\pi} \int_0^\infty f(x) \cos \lambda x\, dx \\ &= \frac{2}{\pi} \int_0^a \cos \lambda x\, dx \\ &= \frac{2}{\pi\lambda} \sin \lambda a. \end{aligned}$$

Therefore,

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda a \cos \lambda x \sinh\left(y\sqrt{\lambda^2 + b}\right)}{\lambda \sinh \sqrt{\lambda^2 + b}}\, d\lambda$$

for $0 < x < \infty, \quad 0 < y < 1$.

Question 1. Find the solution of the **exterior Dirichlet problem for a disk**, that is find a bounded solution to the problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \qquad a<r<\infty, \quad -\pi<\theta<\pi$$

$$u(r,\pi)=u(r,-\pi) \qquad a<r<\infty$$

$$\frac{\partial u}{\partial \theta}(r,\pi)=\frac{\partial u}{\partial \theta}(r,-\pi) \quad a<r<\infty$$

$$u(a,\theta)=f(\theta) \qquad -\pi<\theta<\pi.$$

Solution: A solution to Laplace's equation in polar coordinates which satisfies the periodicity conditions is given by

$$u(r,\theta)=A_0+B_0\log r+\sum_{n=1}^{\infty}\left\{r^n\left(A_n\cos n\theta+B_n\sin n\theta\right)+\frac{1}{r^n}\left(C_n\cos n\theta+D_n\sin n\theta\right)\right\},$$

and in order to satisfy the boundedness condition we need $B_0=A_n=B_n=0$, for $n=1,2,3,\ldots$, so that

$$u(r,\theta)=A_0+\sum_{n=1}^{\infty}\frac{1}{r^n}\left(C_n\cos n\theta+D_n\sin n\theta\right).$$

Now, when $r=a$ we have

$$f(\theta)=u(a,\theta)=A_0+\sum_{n=1}^{\infty}\frac{1}{a^n}\left(C_n\cos n\theta+D_n\sin n\theta\right),$$

where

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\,d\phi,$$

$$C_n=\frac{a^n}{\pi}\int_{-\pi}^{\pi}f(\phi)\cos n\phi\,d\phi,$$

$$D_n=\frac{a^n}{\pi}\int_{-\pi}^{\pi}f(\phi)\sin n\phi\,d\phi$$

for $n=1,2,3\ldots$.

Therefore

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\,d\phi+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\int_{-\pi}^{\pi}f(\phi)\left\{\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\right\}d\phi,$$

that is,

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\left\{1+2\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\theta-\phi)\right\}d\phi.$$

Now let $z=\frac{a}{r}e^{i(\theta-\phi)}$, then

$$z^n=\left(\frac{a}{r}\right)^n e^{in(\theta-\phi)}=\left(\frac{a}{r}\right)^n\left[\cos n(\theta-\phi)+i\sin n(\theta-\phi)\right],$$

and

$$1+2\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\theta-\phi)=\mathrm{Re}\left(1+2\sum_{n=1}^{\infty}z^n\right).$$

Since $|z| = \dfrac{a}{r} < 1$, then

$$1 + 2\sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos n(\theta - \phi) = \text{Re}\left(1 + \frac{2z}{1-z}\right) = \text{Re}\left(\frac{1+z}{1-z}\right) = \frac{r^2 - a^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}.$$

The solution to the exterior Dirichlet problem for the disk is therefore

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(r^2 - a^2)\, f(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi,$$

for $a < r < \infty, \quad -\pi < \theta < \pi.$