Math 300
Solutions to Midterm Examination I

Date: Wednesday June 6, 2018 Time: 50 Minutes

Instructor: I. E. Leonard (Section C1)

The Fourier Series Question.

(a) Show that

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

for $-\pi \le x \le \pi$.

(b) Show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Solution:

(a) First note that $|-x| = |x|$ for all $x \in [-\pi, \pi]$, so that $|x|$ is an even function, therefore

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin nx\,dx = 0$$

for all $n \ge 1$. The Fourier series for $|x|$ can be written

$$|x| \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2},$$

and for $n \ge 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\frac{x\sin nx}{n}\bigg|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin nx\,dx = \frac{2}{n^2\pi}\cos nx\bigg|_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{for } n \text{ even} \\ -\dfrac{4}{n^2\pi}, & \text{for } n \text{ odd.} \end{cases}$$

From Dirichlet's theorem, since the periodic extension of $|x|$ is continuous everywhere, then

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

for $x \in [-\pi, \pi]$.

(b) Setting $x = 0$, we have

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},$$

so that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

The Boundary Value Problem Question.

Find the eigenvalues and eigenfunctions of the boundary value problem

$$\begin{aligned}
&\phi'' + \lambda\phi = 0, \quad 0 < x < a \\
&\phi(0) = 0, \\
&\phi'(a) = 0.
\end{aligned}$$

SOLUTION: We consider three cases:

case (i): $\lambda = 0$

The differential equation is

$$\phi'' = 0,$$

with general solution

$$\phi(x) = Ax + B.$$

The boundary condition $\phi(0) = 0$ implies that $B = 0$, and the boundary condition $\phi'(a) = 0$ implies that $A = 0$, so there are no nontrivial solutions in this case.

case (ii): $\lambda < 0$

Here we write $\lambda = -\mu^2$, where $\mu \neq 0$.

The differential equation is

$$\phi'' - \mu^2\phi = 0,$$

with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x.$$

The boundary condition $\phi(0) = 0$ implies that $A = 0$, and the boundary condition $\phi'(a) = 0$ implies that $\mu B\cosh\mu a = 0$, and since $\mu \neq 0$ and $\cosh\mu a \neq 0$, then we must have $B = 0$. Thus, there are no nontrivial solutions in this case.

case (iii): $\lambda > 0$

Here we write $\lambda = \mu^2$, where $\mu \neq 0$.

The differential equation is

$$\phi'' + \mu^2\phi = 0,$$

with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x.$$

The boundary condition $\phi(0) = 0$ implies that $A = 0$, and the boundary condition $\phi'(a) = 0$ implies that $\mu B\cos\mu a = 0$, and since $\mu \neq 0$, in order to get a nontrivial solution we need $\cos\mu a = 0$, that is, $\mu a = \dfrac{(2n-1)\pi}{2}$.

Therefore, the eigenvalues are

$$\lambda_n = \mu_n^2 = \frac{(2n-1)^2\pi^2}{4a^2}$$

for $n \geq 1$, and the corresponding eigenfunctions are

$$\phi_n(x) = \sin\tfrac{(2n-1)\pi x}{2a}$$

for $n \geq 1$.

The Equilibrium Question.

State and solve the steady-state problem corresponding to

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma^2 u, \quad 0 < x < a,\ t > 0$$
$$u(0,t) = T_0,$$
$$u(a,t) = T_1,$$

where k and γ are constant.

Solution: The steady-state $\overline{u}$ solution does not depend on t, so that

$$0 = \frac{1}{k}\frac{\partial \overline{u}}{\partial t} = \frac{\partial^2 \overline{u}}{\partial x^2} - \gamma^2 \overline{u},$$

and the partial derivatives with respect to x become total derivatives, so that $\overline{u}$ satisfies the following boundary value problem:

$$\frac{d^2\overline{u}}{dx^2} - \gamma^2\overline{u} = 0, \quad 0 < x < a$$
$$\overline{u}(0) = T_0,$$
$$\overline{u}(a) = T_1.$$

In order to find the general solution to this linear ordinary differential equation, we need two linearly independent solutions. In order to make the evaluation of the constants as easy as possible, we choose these solutions so that one them vanishes at $x = 0$, while the other vanishes at $x = a$. The general solution is

$$\overline{u}(x) = A\sinh\gamma x + B\sinh\gamma(a - x).$$

The boundary condition $\overline{u}(0) = T_0$ implies that

$$B = \frac{T_0}{\sinh\gamma a},$$

while the boundary condition $\overline{u}(a) = T_1$ implies that

$$A = \frac{T_1}{\sinh\gamma a}.$$

If $\gamma \neq 0$, the steady-state solution is

$$\overline{u}(x) = \frac{T_1\sinh\gamma x}{\sinh\gamma a} + \frac{T_0\sinh\gamma(a - x)}{\sinh\gamma a}$$

for $0 < x < a$.

If $\gamma = 0$, the solution is $\overline{u}(x) = Ax + B$, and since $\overline{u}(0) = T_0$, then $B = T_0$, while since $\overline{u}(a) = T_1$, we have $T_1 = \overline{u}(a) = Aa + T_0$, and $A = (T_1 - T_0)/a$. Therefore,

$$\overline{u}(x) = \frac{(T_1 - T_0)x}{a} + T_0 = \frac{T_1 x}{a} + \frac{T_0(a - x)}{a}$$

if $\gamma = 0$.

I cannot change the marking scheme without the consent of all the students in the class. Since there were only 19 out of the total of 78 present in today's lecture, I am going to have to grade the exams as usual.

Ed Leonard