Math 300 Winter 2018
Advanced Boundary Value Problems I

Differentiation and Integration of Fourier Series

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Differentiation of Fourier Series

In this note we summarize the results on the term-by-term differentiation of Fourier series, Fourier cosine series and Fourier sine series, all of which follow from Fourier's theorem.

We also clarify the ambiguous statements on term-by-term differentiation of Fourier series on page 118 of the text (as well as correct a misprint).

The results that the author intended to convey are given below as a series of four lemmas.

Fourier Series

Lemma 1. Let f be a function such that

(i) f is continuous on the interval $-\pi \leq x \leq \pi$

(ii) $f(-\pi) = f(\pi)$

(iii) f' is piecewise continuous on the interval $-\pi < x < \pi$.

The Fourier series representation of

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx,$$

for $n \geq 1$, is differentiable at each point x_0 with $-\pi < x_0 < \pi$ at which $f''(x_0)$ exists, and

$$f'(x_0) = \sum_{n=1}^{\infty} n(-a_n \sin nx_0 + b_n \cos nx_0). \tag{$*$}$$

At a point x_0 with $-\pi < x_0 < \pi$ at which $f''(x_0)$ does not exist, but where f' has one-sided derivatives, then the series in $(*)$ converges to

$$\frac{f'(x_0^+) + f'(x_0^-)}{2}.$$

Note: Conditions (i), (ii) and (iii) are needed to ensure that the periodic extension of f is continuous everywhere, and that the Fourier series of f converges to this extension at each point. Condition (iii) is also needed to ensure that the Fourier series for f' converges.

In the text the assumption that f' is piecewise smooth says that f' and f'' are both piecewise continuous, and then Dirichlet's theorem applied to f' gives us the result.

Fourier Cosine Series

Lemma 2. Let f be a function such that

(i) f is continuous on the interval $0 \le x \le \pi$

(ii) f' is piecewise continuous on the interval $0 < x < \pi$.

The Fourier cosine series representation of

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx,$$

for $n \ge 1$, is differentiable at each point x_0 with $0 < x_0 < \pi$ at which $f''(x_0)$ exists, and

$$f'(x_0) = -\sum_{n=1}^{\infty} n a_n \sin nx_0. \qquad (**)$$

At a point x_0 with $0 < x_0 < \pi$ at which $f''(x_0)$ does not exist, but where f' has one-sided derivatives, then the series in $(**)$ converges to

$$\frac{f'(x_0^+) + f'(x_0^-)}{2}.$$

Note: Conditions (i) and (ii) are needed to ensure that the periodic extension of f_{even} is continuous everywhere, and that the Fourier series of f_{even} converges to this extension at each point. Condition (ii) is also needed to ensure that the Fourier sine series for f' converges.

Fourier Sine Series

Lemma 3. Let f be a function such that

(i) f is continuous on the interval $0 \le x \le \pi$

(ii) $f(0) = f(\pi) = 0$

(iii) f' is piecewise continuous on the interval $0 < x < \pi$.

The Fourier sine series representation of

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx,$$

for $n \ge 1$, is differentiable at each point x_0 with $0 < x_0 < \pi$ at which $f''(x_0)$ exists, and

$$f'(x_0) = \sum_{n=1}^{\infty} n b_n \cos nx_0. \qquad (***)$$

At a point x_0 with $0 < x_0 < \pi$ at which $f''(x_0)$ does not exist, but where f' has one-sided derivatives, then the series in $(***)$ converges to

$$\frac{f'(x_0^+) + f'(x_0^-)}{2}.$$

Note: Conditions (i), (ii) and (iii) are needed to ensure that the periodic extension of f_{odd} is continuous everywhere, and that the Fourier series of f_{odd} converges to this extension at each point. Condition (iii) is also needed to ensure that the Fourier cosine series for f' converges.

Integration of Fourier Series

Lemma 4. Let f be a function that is piecewise continuous on the interval $-\pi < x < \pi$, if f is represented by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{+}$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,$$

for $n \geq 1$, regardless of whether the series $(+)$ converges or not, we can integrate term-by-term and obtain

$$\int_{-\pi}^{x} f(t)\,dt = a_0(x+\pi) + \sum_{n=1}^{\infty}\frac{1}{n}\{a_n \sin nx - b_n[\cos nx - (-1)^n]\}$$

for $-\pi \leq x \leq \pi$.