Math 300 Spring-Summer 2018

Advanced Boundary Value Problems I

Derivation of the One-Dimensional Wave Equation

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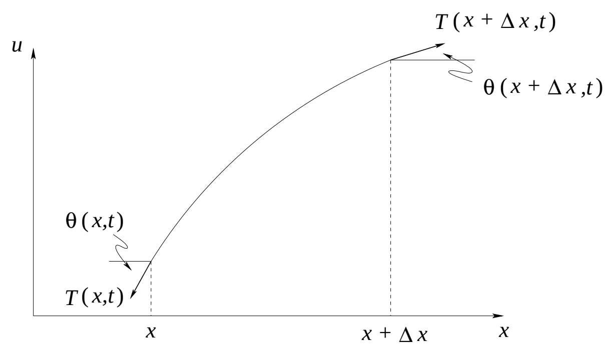
In this note, we derive the one-dimensional wave equation for small vertical displacements of a perfectly elastic string of length L.

We assume the string is stretched under tension and fastened at two points A and$B$, and we let x denote the distance from the end A toward the end b, and let t denote the time.

At time $t = 0$, the string is set in motion, and we let $u(x,t)$ denote the vertical displacement of the string at position x, at time t.

We assume the string is flexible, so there is no resistance to bending, and we let $T(x,t)$ denote the tension in the string at position x, at time t.



Applying Newton's second law to the small portion of the string between x and $x + \Delta x$, if ρ is the mass per unit length, we have

$$\frac{\partial}{\partial t}\left(\rho \Delta x \frac{\partial u}{\partial t}\right) \approx T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) - \alpha\frac{\partial u}{\partial t}\Delta x - \beta u \Delta x + Q(x,t)\Delta x,$$

where $\alpha > 0$ and $\beta > 0$.

The term $-\alpha\dfrac{\partial u}{\partial t}$ represents any resistance force per unit length, the term $-\beta u$ represents any restoring force per unit length, and the term $Q(x,t)$ represents any external forces (such as gravity) per unit length.

Dividing by Δx and letting $\Delta x \to 0$, we get equality in the limit, so that u satisfies

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(T(x,t)\sin\theta(x,t)\right) - \alpha\frac{\partial u}{\partial t} - \beta u(x,t) + Q(x,t) \qquad (*)$$

for $t \geq 0,\ 0 < x < L$.

Now we make some simplifying assumptions.

• For *small vertical displacements* then we have the approximation

$$\sin\theta(x,t) \approx \tan\theta(x,t) = \frac{\partial u}{\partial x}$$

and the PDE $(*)$ becomes

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(T(x,t)\frac{\partial u}{\partial x}\right) - \alpha\frac{\partial u}{\partial t} - \beta u + Q(x,t) \qquad (**)$$

for $t \geq 0,\ 0 < x < L$.

• If the string is *perfectly elastic* then $T \approx$ constant $= T_0$, the initial tension.

• If the string is made from a *uniform material*, then $\rho(x) =$ constant, and the PDE $(**)$ becomes

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial t} - \beta u + Q(x,t) \qquad (***)$$

for $t \geq 0,\ 0 < x < L$.

• If the *tension T is large compared to $Q(x,t)$*, we may neglect $Q(x,t)$ and the PDE $(***)$ becomes

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial t} - \beta u \qquad (\dagger)$$

for $t \geq 0,\ 0 < x < L$. This equation is called the **Telegrapher's Equation**, and models, among other things electromagnetic wave transmission in a wire. In this context, it usually is written as

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + (RC + LG)\frac{\partial u}{\partial t} + RGu$$

where u is either the magnitude E of the voltage at any point in the wire, or the current i at any point in the wire. Here, R is the series resistence per unit length, L is the inductance per unit length (not to confused with the length of the wire), C is the capactitance per unit length, and G conductance per unit length.

• If there are *no frictional forces* and *no restoring forces*, then the PDE $(\dagger)$ becomes

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2},$$

that is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad (\dagger\dagger)$$

for $t \geq 0,\ 0 < x < L$, where $c^2 = \dfrac{T_0}{\rho}$, and c is the velocity of wave propagation along the string. This is the **One-Dimensional Wave Equation**, and models sound waves, water waves, vibrations in solids, longitudinal or torsional vibrations in a rod, among other things.

From our rule of thumb for side conditions, we need two boundary conditions and two initial conditions.

The initial conditions usually take the form of

(i) the **initial displacement**

$$u(x,0) = f(x), \quad 0 \leq x \leq L,$$

and

(ii) the **initial velocity**

$$v(x,0) = \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \leq x \leq L.$$

Typical boundary conditions are of the same form given in the discussion of the one-dimensional heat equation.

1ST KIND: Dirichlet Conditions

$$\begin{aligned} u(0,t) &= g_1(t), \\ u(L,t) &= g_2(t) \end{aligned}$$

for $t \geq 0$. Here the ends move with time in a vertical motion only.

For homogeneous Dirichlet conditions,

$$\begin{aligned} u(0,t) &= 0, \\ u(L,t) &= 0 \end{aligned}$$

for $t \geq 0$, the ends of the string are tied down.

2ND KIND: Neumann Conditions

Here the tensile force $T\dfrac{\partial u}{\partial x}$ is specified at the ends.

$$\begin{aligned} T(0,t)\frac{\partial u}{\partial x}(0,t) &= g_1(t), \\ T(L,t)\frac{\partial u}{\partial x}(L,t) &= g_2(t) \end{aligned}$$

for $t \geq 0$.

For constant tensile force, we have homogeneous Neumann conditions,

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0, \\ \frac{\partial u}{\partial x}(L,t) &= 0 \end{aligned}$$

for $t \geq 0$. These conditions can be achieved, for example, by attaching the ends of the string to a frictionless sleeve which moves vertically.

3RD KIND: Robin Conditions

Here the conditions describe some type of elastic attachment at both ends

$$\begin{aligned} T(0,t)\frac{\partial u}{\partial x}(0,t) &= k_1 u(0,t), \\ T(L,t)\frac{\partial u}{\partial x}(L,t) &= -k_2 u(L,t) \end{aligned}$$

where the spring constants are $k_1 > 0$ and $k_2 > 0$, and both springs have the other end fixed.

Or, the other ends of the springs can move vertically

$$\begin{aligned} T(0,t)\frac{\partial u}{\partial x}(0,t) &= k_1[u(0,t) - d_1(t)], \\ T(L,t)\frac{\partial u}{\partial x}(L,t) &= -k_2[u(L,t) - d_2(t)] \end{aligned}$$

for $t \geq 0$.

Boundary Value Problems with Periodicity Conditions

Sometimes in the statement of an Initial Value – Boundary Value Problem, the side conditions take the form of *periodicity conditions* instead of boundary conditions or initial conditions, in order to maintain continuity of the solution across artifical boundaries, as with problems in planar polar coordinates.

Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

subject to the periodicity conditions

$$\phi(0) = \phi(2\pi) \qquad \text{and} \qquad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

SOLUTION:Again, we consider three cases.

case 1: If $\lambda = 0$, then the equation is $\phi'' = 0$ with general solution $\phi(x) = Ax + B$. From the first periodicity condition $\phi(0) = \phi(2\pi)$ we have

$$\phi(0) = A \cdot 0 + B = A \cdot 2\pi + B,$$

so that $2\pi A = 0$, and $A = 0$. The solution is now

$$\phi(x) = B, \quad 0 \le x \le 2\pi.$$

The second periodicity condition $\phi'(0) = \phi(2\pi)$ holds automatically, since

$$\phi'(0) = 0 = \phi'(2\pi).$$

Therefore $\lambda_0 = 0$ is an eigenvalue with corresponding eigenfunction

$$\phi_0(x) = 1, \quad 0 \le x \le 2\pi.$$

case 2: If $\lambda < 0$, then $\lambda = -\mu^2$ where $\mu \neq 0$. The differential equation is $\phi'' - \mu^2\phi = 0$ with general solution

$$\phi(x) = A\cosh\mu x + B\sinh\mu x, \quad 0 \le x \le 2\pi.$$

From the first periodicity condition

$$\phi(0) = A = A\cosh 2\pi\mu + B\sinh 2\pi\mu = \phi(2\pi),$$

while from the second periodicity condition

$$\phi'(0) = \mu B = \mu A\sinh 2\pi\mu + \mu B\cosh 2\pi\mu = \phi'(2\pi).$$

We have the homogeneous system of linear equations for A and B

$$\begin{aligned}(\cosh 2\pi\mu - 1)\,A + \sinh 2\pi\mu\,B &= 0\\ \sinh 2\pi\mu\,A + (\cosh 2\pi\mu - 1)\,B &= 0,\end{aligned}$$

and the determinant of the coefficient matrix is

$$\begin{vmatrix}\cosh 2\pi\mu - 1 & \sinh 2\pi\mu\\ \sinh 2\pi\mu & \cosh 2\pi\mu - 1\end{vmatrix} = 2(1 - \cosh 2\pi\mu) = -4\sinh^2\pi\mu \neq 0$$

since $\pi\mu \neq 0$, and this system has only the trivial solution $A = B = 0$. In this case the boundary value problem has only the trivial solution $\phi(x) = 0$ for $0 \le x \le 2\pi$.

<u>case 3</u>: If $\lambda > 0$, then $\lambda = \mu^2$ where $\mu \neq 0$, and the differential equation is $\phi'' + \mu^2 \phi = 0$ with general solution

$$\phi(x) = A\cos\mu x + B\sin\mu x, \quad 0 \leq x \leq 2\pi.$$

From the first periodicity condition

$$\phi(0) = A = A\cos 2\pi\mu + B\sin 2\pi\mu = \phi(2\pi),$$

while from the second periodicity condition

$$\phi'(0) = \mu B = -\mu A\sin 2\pi\mu + \mu B\cos 2\pi\mu = \phi'(2\pi).$$

We have the homogeneous system of linear equations for A and B

$$\begin{aligned}(1 - \cos 2\pi\mu)\, A + \sin 2\pi\mu\, B &= 0\\ -\sin 2\pi\mu\, A + (1 - \cosh 2\pi\mu)\, B &= 0,\end{aligned}$$

and the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 - \cos 2\pi\mu & \sin 2\pi\mu \\ -\sin 2\pi\mu & 1 - \cos 2\pi\mu \end{vmatrix} = 2(1 - \cos 2\pi\mu) = 4\sin^2 \pi\mu$$

and this system has a nontrivial solution if and only if this determinant is zero, that is, if and only if $\sin^2 \pi\mu = 0$, that is if and only if $\pi\mu = n\pi$ for some integer n.

In this case the boundary value problem has a nontrivial solution if and only if $\mu = n$ for some integer n. The eigenvalues are

$$\lambda_n = \mu_n^2 = n^2,$$

and the corresponding eigenfunctions are

$$\phi_n(x) = A_n\cos nx + B_n\sin nx, \quad 0 \leq x \leq 2\pi$$

for $n \geq 1$.

Note that for each eigenvalue $\lambda_n = n$, $n \geq 1$, we have two corresponding eigenfunctions; namely, $\cos nx$ and $\sin nx$.