Math 300 Winter 2018
Advanced Boundary Value Problems I

Bessel's Inequality

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Department of Mathematical and Statistical Sciences
University of Alberta

The following inequality is known as **Bessel's Inequality** and can be found on page 66 of the text.

Theorem. Let $f(x)$ be piecewise smooth on the interval $[-\pi, \pi]$, and let the Fourier series of f be

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt, \quad n \geq 1$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt, \quad n \geq 1$$

then

$$a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)^2\, dt.$$

Proof. Let $S_n(x)$ be the n^{th} partial sum of the Fourier series, that is,

$$S_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Using the orthogonality, we have

$$\int_{-\pi}^{\pi} a_0\, S_n(t)\, dt = 2\pi a_0^2 = a_0 \int_{-\pi}^{\pi} f(t)\, dt$$

$$\int_{-\pi}^{\pi} a_k \cos kt\, S_n(t)\, dt = \pi a_k^2 = a_k \int_{-\pi}^{\pi} f(t) \cos kt\, dt$$

$$\int_{-\pi}^{\pi} b_k \sin kt\, S_n(t)\, dt = \pi b_k^2 = b_k \int_{-\pi}^{\pi} f(t) \sin kt\, dt$$

for $k = 1, 2, \ldots, n$.

Adding the equations on the right-hand side, we have

$$\int_{-\pi}^{\pi} f(t)\, S_n(t)\, dt = 2\pi a_0^2 + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2) \qquad (*)$$

and adding the equations on the left-hand side, we have

$$\int_{-\pi}^{\pi} S_n(t)^2\, dt = 2\pi a_0^2 + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2) \qquad (**)$$

for $n \geq 1$.

Now,

$$
\begin{aligned}
0 &\le \int_{-\pi}^{\pi} \left[f(t) - S_n(t)\right]^2 \, dt \\
&= \int_{-\pi}^{\pi} f(t)^2 \, dt - 2\int_{-\pi}^{\pi} f(t)\, S_n(t)\, dt + \int_{-\pi}^{\pi} S_n(t)^2 \, dt \\
&= \int_{-\pi}^{\pi} f(t)^2 \, dt - \int_{-\pi}^{\pi} S_n(t)^2 \, dt \\
&= \int_{-\pi}^{\pi} f(t)^2 \, dt - \left[2\pi a_0^2 + \pi \sum_{k=1}^{n} (a_k^2 + b_k^2)\right],
\end{aligned}
$$

so that

$$
a_0^2 + \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)^2 \, dt
$$

for all $n \ge 1$.

Letting $n \to \infty$, we have

$$
a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)^2 \, dt,
$$

and the series $a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$ converges. □

Note: We only needed the function f to be piecewise continuous on the interval $[-\pi, \pi]$ so that the integral

$$
\int_{-\pi}^{\pi} f(t)^2 \, dt
$$

exists and is finite.

Also, if we assume that f is piecewise smooth on the interval $[-\pi, \pi]$, and that $f(\pi) = f(-\pi)$, then we can use Bessel's Inequality to show that the Fourier series for f converges at each point of the interval. It is a little more difficult to show that it converges to f at the points of continuity of f.

Corollary. If f is piecewise smooth on the interval $[-\pi, \pi]$, and $f(\pi) = f(-\pi)$, then the Fourier series

$$
a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
$$

converges absolutely and uniformly on $[-\pi, \pi]$.

Proof. Since f' is piecewise continuous on the interval $[-\pi, \pi]$ then the Fourier coefficients of f' exist and the Fourier series of f' can be written

$$
f'(x) \sim \alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx),
$$

where

$$
\begin{aligned}
\alpha_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)\, dt \\
\alpha_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t) \cos nt \, dt, \\
\beta_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t) \sin nt \, dt,
\end{aligned}
$$

for $n \ge 1$.

Now,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(t)\, dt = \frac{1}{2\pi} \left[f(\pi) - f(-\pi)\right] = 0,$$

and integrating by parts, we have

$$\alpha_n = n\, b_n \qquad \text{and} \qquad \beta_n = -n\, a_n$$

for $n \geq 1$.

From the Cauchy-Schwarz inequality and Bessel's inequality applied to f', we have

$$\begin{aligned}
\sum_{n=1}^{N} \sqrt{a_n^2 + b_n^2} &= \sum_{n=1}^{N} \frac{1}{n} \sqrt{\alpha_n^2 + \beta_n^2} \\
&\leq \left(\sum_{n=1}^{N} \frac{1}{n^2} \right) \left(\sum_{n=1}^{N} (\alpha_n^2 + \beta_n^2) \right) \\
&\leq \frac{\pi^2}{6} \cdot \sum_{n=1}^{N} (\alpha_n^2 + \beta_n^2) \\
&\leq \frac{\pi}{12} \cdot \int_{-\pi}^{\pi} f'(t)^2\, dt,
\end{aligned}$$

for all $N \geq 1$, and so $\sum\limits_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ converges.

For each $n \geq 1$, we have

$$|a_n| \leq \sqrt{a_n^2 + b_n^2} \qquad \text{and} \qquad |b_n| \leq \sqrt{a_n^2 + b_n^2},$$

so that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| a_n \cos nt + b_n \sin nt \right| &\leq \sum_{n=1}^{\infty} (|a_n| + |b_n|) \\
&\leq 2 \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \\
&\leq \frac{\pi}{6} \int_{-\pi}^{\pi} f'(t)^2\, dt < \infty,
\end{aligned}$$

and the Fourier series for f converges absolutely and uniformly on the interval $[-\pi, \pi]$. □