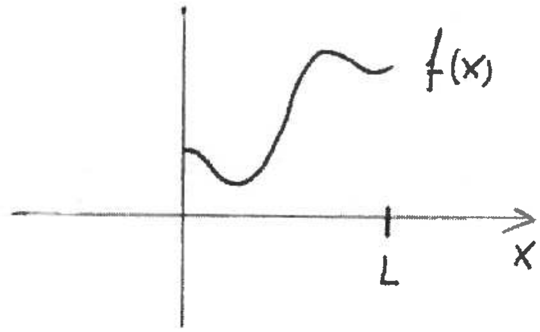
(4.4) Even and odd extensions

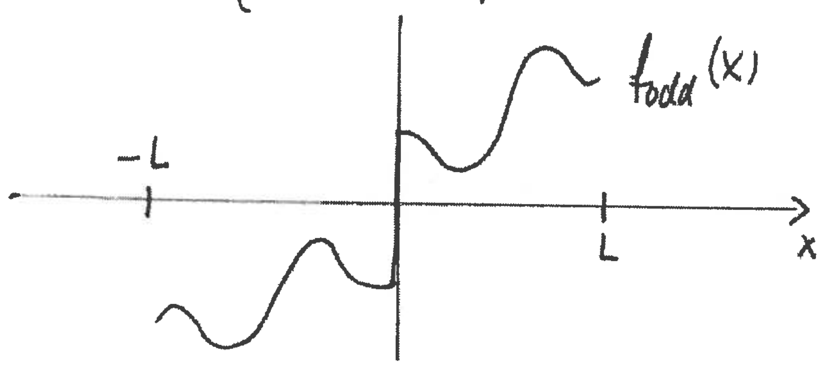
Assume a function $f(x)$ is only given on $[0, L]$.
How to find the Fourier-series?



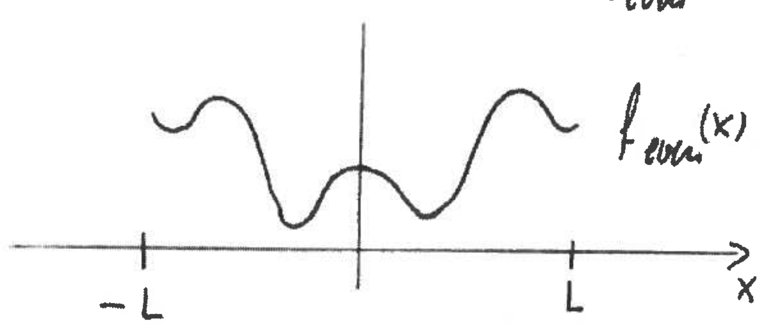
There are two ways to find the Fourier-series

(1) We use the <u>odd extension</u>

$$f_{odd}(x) = \begin{cases} -f(-x) & \text{for } -L < x < 0 \\ f(x) & \text{for } 0 < x < L \end{cases}$$

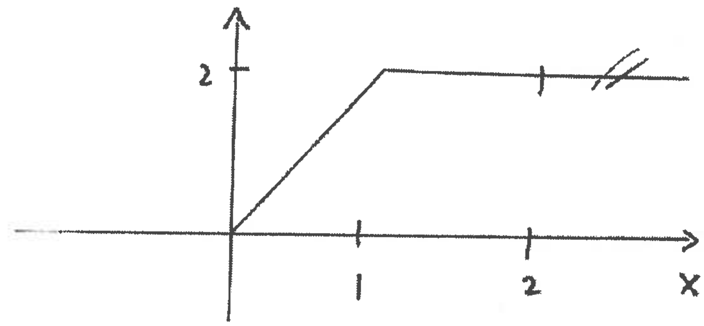


or (2) We use the <u>even extension</u> $f_{even}(x) = \begin{cases} f(-x), & -L < x < 0 \\ f(x), & 0 < x < L \end{cases}$



Example: Find the Fourier series of

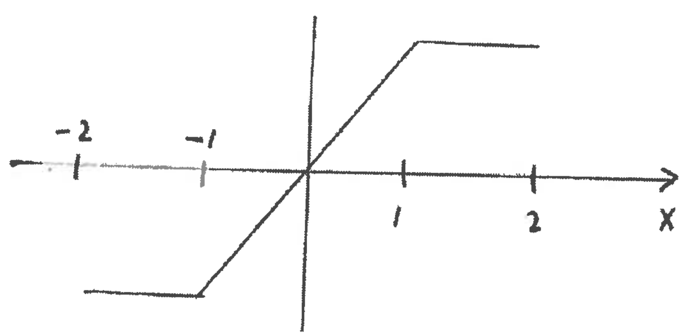
$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases} \quad \text{on } [0,2].$$



Solution 1: The odd extension:

$$f_{odd}(x) = \begin{cases} -f(-x), & -2 < x < 0 \\ f(x), & 0 < x < 2 \end{cases} = \begin{cases} -2, & -2 < x < -1 \\ 2x, & -1 < x < 0 \\ 2x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$$

$$= \begin{cases} -2, & -2 < x < -1 \\ 2x, & -1 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$$



1) f_{odd} is odd $\Rightarrow$ $a_0 = 0, \quad a_n = 0$

2) $$b_n = \frac{1}{2} \int_{-2}^{2} f_{odd}(x) \sin\left(\frac{n\pi x}{2}\right) dx = \int_{0}^{2} f_{odd}(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$b_n = \int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 2x \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 2 \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= -2x \frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\Big]_0^1 + 2\int_0^1 \frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) dx$$

$$+2 \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\Big]_1^2$$

$$= \frac{-4}{n\pi} \cos\left(\frac{n}{2}\pi\right) + \frac{4}{n\pi} \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big]_0^1$$

$$- \frac{4}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n}{2}\pi\right)\right)$$

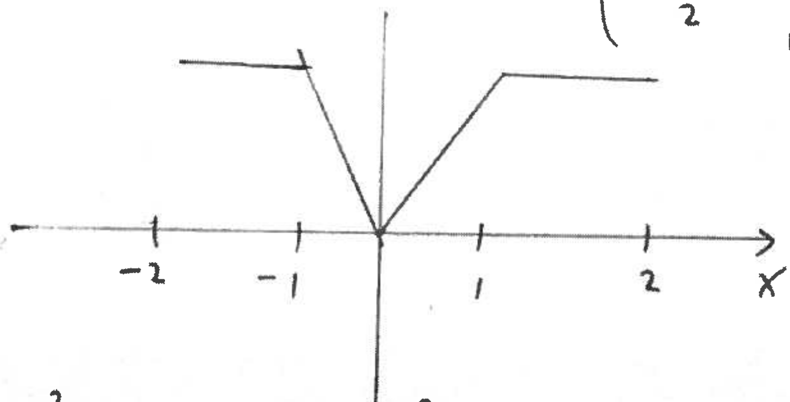
$$= \frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi} \cos(n\pi)$$

$$\Rightarrow \boxed{f(x) = \sum_{n=1}^{\infty} \left(\frac{8}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi} \cos(n\pi)\right) \sin\left(\frac{n\pi x}{2}\right)}$$

Fourier - sine - series of $f(x)$ on $[0,2]$.

<u>Solution 2:</u> The even extension

$$f_{even}(x) = \begin{cases} f(-x), & -2<x<0 \\ f(x), & 0<x<2 \end{cases} = \begin{cases} 2, & -2<x<-1 \\ -2x, & -1<x<0 \\ 2x, & 0<x<1 \\ 2, & 1<x<2 \end{cases}$$



1) $$a_0 = \frac{1}{4}\int_{-2}^{2} f_{even}(x)\,dx = \frac{1}{2}\int_{0}^{2} f_{even}(x)\,dx = \frac{1}{2}\int_{0}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{0}^{1} 2x\,dx + \frac{1}{2}\int_{1}^{2} 2\,dx = \frac{1}{2} + 1 = \frac{3}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f_{even}(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_{0}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{1} 2x\cos\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2} 2\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= 2x\,\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg]_{0}^{1} - 2\int_{0}^{1}\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$+ 2\,\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg]_{1}^{2}$$

$$= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\,\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big]_0^1 + \frac{4}{n\pi}\sin(n\pi)$$

$$- \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{8}{(n\pi)^2}\left(\cos\left(\frac{n}{2}\pi\right) - 1\right)$$

$\cancel{\Rightarrow f(x) = \frac{3}{2}}$

2) $b_n = \frac{1}{2}\int_{-2}^{2} f_{even}(x)\sin\left(\frac{n\pi x}{2}\right)dx = 0$

$$\Rightarrow \boxed{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2}\left(\cos\left(\frac{n}{2}\pi\right) - 1\right)\cos\left(\frac{n\pi x}{2}\right)}$$

<u>Fourier</u> - <u>cosine</u> - <u>series</u> of <u>$f(x)$ on $[0,2]$</u>.

<u>Summary</u>: Given $f(x)$ on $[0,L]$, $f \in PWS[0,L]$.

The <u>Fourier - cosine - series</u> of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with
$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

The <u>Fourier-sine-series</u> of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

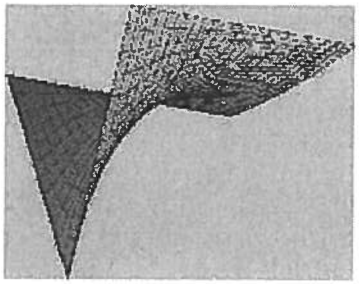
$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

They play different role's as it comes to the solution of PDE's, related to the <u>boundary conditions</u> at 0 and L:
$$\sin\left(\frac{n\pi \cdot 0}{L}\right) = 0, \qquad \sin\left(\frac{n\pi L}{L}\right) = 0$$

$\Rightarrow$ from the sine-series we see $f(0) = 0$, $f(L) = 0$.

Similarily. The cosine-series satisfies

$$\frac{d}{dx} f(0) = 0, \qquad \frac{d}{dx} f(L) = 0.$$



Math 300 Winter 2011

Advanced Boundary Value Problems I

Fourier's

~~Dirichlet's~~ Theorem

This should be in the text!

Wednesday February 2, 2011

Department of Mathematical and Statistical Sciences
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The following theorem on the convergence of Fourier series in known as **Dirichlet's Theorem**, or **Fourier's Theorem**.

Theorem. Let $f(x)$ be piecewise smooth on the interval $[-L, L]$, the Fourier series

$$\mathcal{F}\,\mathcal{S}(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos \tfrac{n\pi t}{L}\,dt, \quad n \geq 1$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin \tfrac{n\pi t}{L}\,dt, \quad n \geq 1$$

has the following properties:

(i) If $f(x)$ is continuous at x_0, where $-L < x_0 < L$, then

$$f(x_0) = \mathcal{F}\,\mathcal{S}(x_0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x_0}{L} + b_n \sin \tfrac{n\pi x_0}{L}\right),$$

that is, the Fourier series converges to $f(x_0)$.

(ii) If $f(x)$ has a jump discontinuity at x_0, where $-L < x_0 < L$, then

$$\frac{f(x_0^+) + f(x_0^-)}{2} = \mathcal{F}\,\mathcal{S}(x_0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x_0}{L} + b_n \sin \tfrac{n\pi x_0}{L}\right),$$

that is, the Fourier series converges to the **average** or the **mean** of the jump.

(iii) At the endpoints $x_0 = \pm L$, the Fourier series converges to

$$\frac{f(-L^+) + f(L^-)}{2}.$$

Usually we write

$$f(x) \sim \mathcal{F}\,\mathcal{S}(x_0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x_0}{L} + b_n \sin \tfrac{n\pi x_0}{L}\right),$$

and say that $f(x)$ is *represented by its Fourier series* on the interval $[-L, L]$.

Note: The Fourier series $\mathcal{F}\,\mathcal{S}(x)$ is periodic with period $2L$, and converges for **all** real numbers x.

For each real number x_0 it converges to the $2L$-periodic extension $\widehat{f}(x_0)$ of f whenever $\widehat{f}$ is continuous at x_0, and to

$$\frac{\widehat{f}(x_0^+) + \widehat{f}(x_0^-)}{2}$$

whenever $\widehat{f}$ has a jump discontinuity at x_0.

This allows us to sketch the graph of the Fourier series $\mathcal{F}\,\mathcal{S}(x)$ once we know what the graph of the function $f(x)$ is, without computing the Fourier series coefficients a_0, a_n and b_n for $n \geq 1$.

Exercise. Sketch the graph of the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -L \leq x < 0 \\ e^{-x} & \text{for} \quad 0 < x < \dfrac{L}{2} \\ 0 & \text{for} \quad \dfrac{L}{2} < x \leq L. \end{cases}$$

Note that the function is undefined for $x = 0$ and $x = \dfrac{L}{2}$, and this does not affect the integrals when we compute the Fourier coefficients. We can sketch the graph of the Fourier series using Dirichlet's theorem, and then go back and redefine the function at the points of discontinuity so that the Fourier series converges to the function for all $x \in [-L, L]$.

Example. Find the Fourier series of the function

$$f(x) = \begin{cases} \sin x, & \text{for } 0 \leq x \leq \pi \\ 0, & \text{for } -\pi \leq x \leq 0. \end{cases}$$

Solution. The Fourier series of $f(x)$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Here,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \int_0^{\pi} \sin x\,dx = -\frac{1}{2\pi} \cos x \Big|_0^{\pi} = \frac{1}{\pi}.$$

For $n = 1$,

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x\,dx = \frac{1}{\pi} \int_0^{\pi} \sin x \cos x\,dx = \frac{1}{2\pi} \sin^2 x \Big|_0^{\pi} = 0.$$

For $n \geq 2$,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{\pi} \sin x \cos nx\,dx \\ &= \frac{1}{2\pi} \int_0^{\pi} \sin(n+1)x\,dx - \frac{1}{2\pi} \int_0^{\pi} \sin(n-1)x\,dx \\ &= -\frac{1}{2\pi(n+1)} \cos(n+1)x \Big|_0^{\pi} + \frac{1}{2\pi(n-1)} \cos(n-1)x \Big|_0^{\pi} \\ &= \frac{(-1)^{n-1}}{2\pi} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) - \frac{1}{2\pi} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\ &= \frac{(-1)^{n-1} - 1}{\pi(n^2 - 1)}. \end{aligned}$$

For $n = 1$,

$$\begin{aligned}
b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{x}{2\pi}\bigg|_0^{\pi} - \frac{1}{4\pi}\sin 2x\bigg|_0^{\pi} \\
&= \frac{1}{2}.
\end{aligned}$$

For $n \geq 2$,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi}\cos(n-1)x\,dx - \frac{1}{2\pi}\int_0^{\pi}\cos(n+1)x\,dx \\
&= \frac{1}{2\pi(n-1)}\sin(n-1)x\bigg|_0^{\pi} - \frac{1}{2\pi(n+1)}\sin(n+1)x\bigg|_0^{\pi} \\
&= 0.
\end{aligned}$$

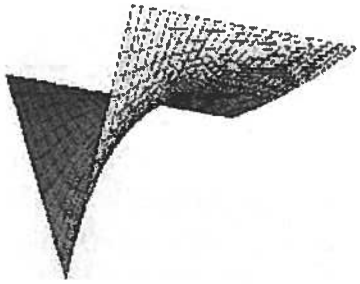
Since the periodic extension of f is continuous everywhere, from Dirichlet's theorem, we have

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}-1}{\pi(n^2-1)}\cos nx,$$

that is,

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n-1)(2n+1)}$$

for $-\pi \leq x \leq \pi$.



Math 300 Winter 2009
Advanced Boundary Value Problems I

Fourier Series Examples

Wednesday January 28, 2009

These should be in the text!

Department of Mathematical and Statistical Sciences
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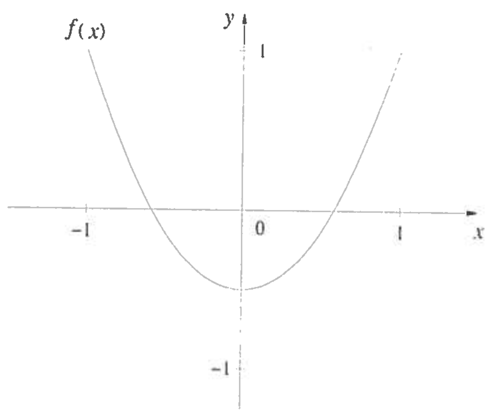
Example 1. Find the Fourier series which represents the function

$$f(x) = x^2 - \frac{1}{2}$$

on the interval $-1 \leq x \leq 1$. Also, use this representation to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Solution. The graph of the function is shown below.



The Fourier coefficients are

$$a_0 = \frac{1}{2}\int_{-1}^{1} \underbrace{\left(x^2 - \frac{1}{2}\right)}_{\text{even}} dx = \int_0^1 \left(x^2 - \frac{1}{2}\right) dx = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6},$$

and

$$a_n = \int_{-1}^{1} \underbrace{\left(x^2 - \frac{1}{2}\right)\cos n\pi x}_{\text{even}}\, dx = 2\int_0^1 x^2 \cos n\pi x\, dx - \int_0^1 \cos n\pi x\, dx = 2\int_0^1 x^2 \cos n\pi x\, dx.$$

Integrating by parts twice, we have

$$a_n = 2\left[\frac{x^2 \sin n\pi x}{n\pi}\Big|_0^1 - \frac{2}{n\pi}\int_0^1 x \sin n\pi x \, dx\right]$$

$$= -\frac{4}{n\pi}\int_0^1 x \sin n\pi x \, dx$$

$$= -\frac{4}{n\pi}\left[-\frac{x \cos n\pi x}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x \, dx\right]$$

$$= \frac{4(-1)^n}{n^2\pi^2},$$

and finally,

$$b_n = \int_{-1}^1 \underbrace{\left(x^2 - \frac{1}{2}\right) \sin n\pi x}_{\text{odd}} \, dx = 0.$$

Therefore, the Fourier series representation of $f(x) = x^2 - \dfrac{1}{2}$ on the interval $-1 \leq x \leq 1$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x) = -\frac{1}{6} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

From Dirichlet's Theorem, the Fourier series converges to $f(x)$ for all $x \in [-1, 1]$, so that we have equality

$$x^2 - \frac{1}{2} = -\frac{1}{6} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

for $-1 \leq x \leq 1$.

Taking $x = 0$ we have

$$-\frac{1}{2} = -\frac{1}{6} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

that is,

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$$

Taking $x = 1$, we have

$$1 - \frac{1}{2} = -\frac{1}{6} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

that is,

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

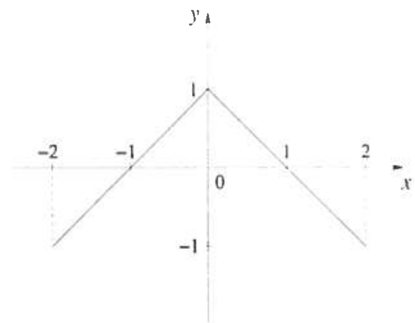
Example 2. Find the Fourier series which represents the function

$$f(x) = \begin{cases} 1 + x & \text{for} \quad -2 < x < 0 \\ 1 - x, & \text{for} \quad 0 < x < 2 \end{cases}$$

on the interval $-2 \leq x \leq 2$. Also, use this representation to show that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

SOLUTION. The graph of the function is shown below. Note that $f(x)$ is an even function on the interval $[-2, 2]$.



The Fourier coefficients are

$$a_0 = \frac{1}{2 \cdot 2} \int_{-2}^{2} f(x)\, dx = \frac{1}{4} \int_{-2}^{0} (1 + x)\, dx + \frac{1}{4} \int_{0}^{2} (1 - x)\, dx = \frac{1}{4} \left[x + \frac{x^2}{2} \right]_{-2}^{0} + \left[x - \frac{x^2}{2} \right]_{0}^{2}.$$

so that,

$$a_0 = \frac{1}{4} [2 - 2] + \frac{1}{4} [2 - 2] = 0,$$

and

$$a_n = \frac{1}{2} \int_{-2}^{2} \underbrace{f(x) \cos \tfrac{n\pi x}{2}}_{\text{even}}\, dx = \int_{0}^{2} (1 - x) \cos \tfrac{n\pi x}{2}\, dx$$

since $f(x) \cos \frac{n\pi x}{2}$ is even. Therefore,

$$\begin{aligned}
a_n &= \int_0^2 \cos \tfrac{n\pi x}{2}\, dx - \int_0^2 x \cos \tfrac{n\pi x}{2}\, dx \\
&= \frac{2}{n\pi} \sin \tfrac{n\pi x}{2} \Big|_0^2 - \frac{2}{n\pi} \sin \tfrac{n\pi x}{2} \Big|_0^2 + \frac{2}{n\pi} \int_0^2 \sin \tfrac{n\pi x}{2}\, dx \\
&= -\frac{4}{n^2\pi^2} \cos \tfrac{n\pi x}{2} \Big|_0^2 \\
&= -\frac{4}{n^2\pi^2} [\cos n\pi - 1] \\
&= \frac{4}{n^2\pi^2} [1 - (-1)^n],
\end{aligned}$$

so that

$$a_n = \frac{4}{n^2\pi^2} [1 - (-1)^n].$$

Finally,

$$b_n = \frac{1}{2} \int_{-2}^{2} \underbrace{f(x) \sin \tfrac{n\pi x}{2}}_{\text{odd}}\, dx = 0,$$

and the Fourier series is

$$f(x) \sim a_0 + \sum_{n=0}^{\infty} \left(a_n \cos \tfrac{n\pi x}{2} + b_n \sin \tfrac{n\pi x}{2} \right) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} [1 - (-1)^n] \cos \tfrac{n\pi x}{2}.$$

Since

$$1-(-1)^n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$$

then $f(x) \sim \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2}$.

From Dirichlet's Theorem, the Fourier series converges to $f(x)$ for all $x \in [-2, 2]$, so that we have equality

$$f(x) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2}$$

for $-2 \leq x \leq 2$.

Taking $x = 0$ we have

$$1 = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

that is,

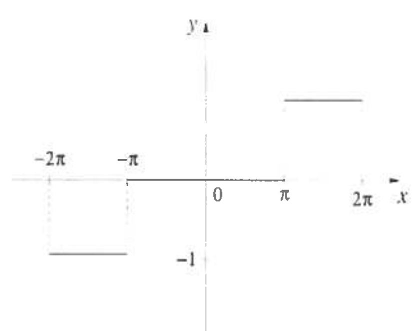
$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Example 3. Find the Fourier series which represents the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -2\pi < x < -\pi \\ 0 & \text{for} \quad -\pi < x < \pi \\ 1 & \text{for} \quad \pi < x < 2\pi \end{cases}$$

on the interval $-2\pi \leq x \leq 2\pi$.

SOLUTION. The graph of the function is shown below.



Note that $f(-x) = -f(x)$, so that the function $f(x)$ is an odd function on the interval $[-2\pi, 2\pi]$, and therefore

$$f(x) \cos \tfrac{n\pi x}{2\pi} = f(x) \cos \tfrac{nx}{2}$$

is also an odd function on the interval $[-2\pi, 2\pi]$, while

$$f(x) \sin \tfrac{n\pi x}{2\pi} = f(x) \sin \tfrac{nx}{2}$$

is an even function on the interval $[-2\pi, 2\pi]$.

Therefore the Fourier coefficients are

$$a_0 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f(x)\,dx = 0,$$

and

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \underbrace{f(x)\cos\tfrac{nx}{2}}_{\text{odd}}\,dx = 0,$$

while

$$b_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \underbrace{f(x)\sin\tfrac{nx}{2}}_{\text{even}}\,dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin\tfrac{nx}{2}\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{2\pi} \sin\tfrac{nx}{2}\,dx = -\frac{2}{n\pi}\cos\tfrac{nx}{2}\Big|_{\pi}^{2\pi}$$

$$= -\frac{2}{n\pi}\left[\cos n\pi - \cos\tfrac{n\pi}{2}\right].$$

However,

$$\cos n\pi - \cos\tfrac{n\pi}{2} = (-1)^n - \cos\tfrac{n\pi}{2},$$

and the Fourier series representing $f(x)$ on the interval $[-2\pi, 2\pi]$ is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\tfrac{n\pi}{2} - (-1)^n\right]\sin\tfrac{nx}{2}.$$

Since

$$\cos\tfrac{n\pi}{2} - (-1)^n = \begin{cases} 0 & \text{for } \; n = 4k \\ -2 & \text{for } \; n = 4k+2 \\ 1 & \text{for } \; n = 4k+1,\ 4k+3 \end{cases}$$

then

$$f(x) \sim \frac{2}{\pi}\left[\sin\tfrac{x}{2} - \frac{1}{2}\sin x + \frac{1}{3}\sin\tfrac{3x}{2} + \frac{1}{5}\sin\tfrac{5x}{2} - \frac{1}{3}\sin 3x + \frac{1}{7}\sin\tfrac{7x}{2} + \frac{1}{9}\sin\tfrac{9x}{2} - \frac{1}{5}\sin 5x + \cdots\right]$$

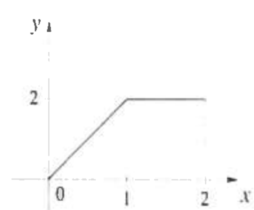
for $-2\pi \le x \le 2\pi$.

In 2.4.1

Example 4. Find the Fourier sine series and the Fourier cosine series, both of which represent the function

$$f(x) = \begin{cases} 2x & \text{for } \; 0 < x < 1 \\ 2 & \text{for} 1 < x < 2 \end{cases}$$

on the interval $0 < x < 2$. The graph of this function is shown below.

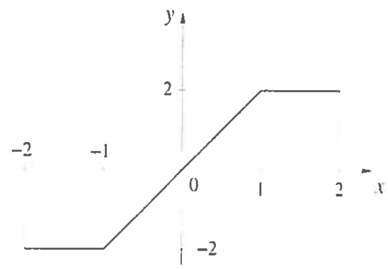


- **Fourier Sine Series**

In order to find the Fourier sine series for $f(x)$ on the interval $[0, 2]$ we consider the odd extension f_{odd} to the interval $[-2, 2]$,

$$f_{\text{odd}}(x) = \begin{cases} -2 & \text{for} \quad -2 < x < -1 \\ 2x & \text{for} \quad -1 < x < 0 \\ 2x & \text{for} \quad 0 < x < 1 \\ 2 & \text{for} \quad 1 < x < 2 \end{cases}$$

whose graph is shown below.



Since f_{odd} is an odd function on $[-2, 2]$, then the Fourier coefficients satisfy $a_0 = 0$ and $a_n = 0$ for $n \geq 1$, while

$$b_n = \frac{1}{2}\int_{-2}^{2} f_{\text{odd}}(x) \sin \tfrac{n\pi x}{2}\, dx = \int_0^2 f(x) \sin \tfrac{n\pi x}{2}\, dx$$

since $f_{\text{odd}} = f$ on the interval $[0, 2]$.

Therefore,

$$\begin{aligned} b_n &= \int_0^1 2x \sin \tfrac{n\pi x}{2}\, dx + \int_1^2 \sin \tfrac{n\pi x}{2}\, dx \\ &= -\frac{4}{n\pi} x \cos \tfrac{n\pi x}{2}\Big|_0^1 + \frac{4}{n\pi}\int_0^1 \cos \tfrac{n\pi x}{2}\, dx - \frac{4}{n\pi} \cos \tfrac{n\pi x}{2}\Big|_1^2 \\ &= -\frac{4}{n\pi} \cos \tfrac{n\pi}{2} + \frac{8}{n^2\pi^2} \sin \tfrac{n\pi x}{2}\Big|_0^1 - \frac{4}{n\pi}\left[\cos n\pi - \cos \tfrac{n\pi}{2}\right] \\ &= \frac{8}{n^2\pi^2} \sin \tfrac{n\pi}{2} - \frac{4}{n\pi} \cos n\pi, \end{aligned}$$

and the Fourier sine series for $f(x)$ on the interval $[0, 2]$ is given by

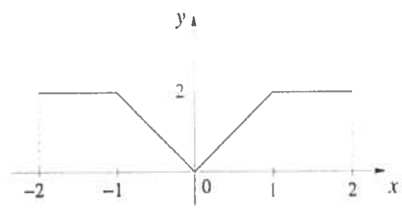
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{2} = \sum_{n=1}^{\infty} \left(\frac{8}{n^2\pi^2} \sin \tfrac{n\pi}{2} - \frac{4}{n\pi} \cos n\pi \right) \sin \tfrac{n\pi x}{2}$$

for $0 < x < 2$.

• **Fourier Cosine Series** In order to find the Fourier cosine series for $f(x)$ on the interval $[0, 2]$ we consider the even extrension f_{even} to the interval $[-2, 2]$,

$$f_{\text{even}}(x) = \begin{cases} 2 & \text{for} \quad -2 < x < -1 \\ -2x & \text{for} \quad -1 < x < 0 \\ 2x & \text{for} \quad 0 < x < 1 \\ 2 & \text{for} \quad 1 < x < 2 \end{cases}$$

whose graph is shown below.



Since f_{even} is an even function on $[-2, 2]$, then the Fourier coefficients satisfy $b_n = 0$ for $n \geq 1$, while

$$\begin{aligned} a_0 &= \frac{1}{4}\int_{-2}^{2} f_{\text{even}}(x)\,dx \\ &= \frac{1}{2}\int_{0}^{2} f_{\text{even}}(x)\,dx = \frac{1}{2}\int_{0}^{2} f(x)\,dx \\ &= \frac{1}{2}\int_{0}^{1} 2x\,dx + \frac{1}{2}\int_{1}^{2} 2\,dx \\ &= \frac{1}{2} + 1 = \frac{3}{2}. \end{aligned}$$

Also,

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f_{\text{even}}(x)\cos\tfrac{n\pi x}{2}\,dx = \int_{0}^{2} f(x)\cos\tfrac{n\pi x}{2}\,dx \\ &= \int_{0}^{1} 2x\cos\tfrac{n\pi x}{2}\,dx + \int_{1}^{2} 2\cos\tfrac{n\pi x}{2}\,dx \\ &= \frac{4}{n\pi}x\sin\tfrac{n\pi x}{2}\Big|_{0}^{1} - \frac{4}{n\pi}\int_{0}^{1}\sin\tfrac{n\pi x}{2}\,dx + \frac{4}{n\pi}\sin\tfrac{n\pi x}{2}\Big|_{1}^{2} \\ &= \frac{4}{n\pi}\sin\tfrac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\tfrac{n\pi x}{2}\Big|_{0}^{1} + \frac{4}{n\pi}\left[\sin n\pi - \sin\tfrac{n\pi}{2}\right] \\ &= \frac{8}{n^2\pi^2}\left[\cos\tfrac{n\pi}{2} - 1\right]. \end{aligned}$$

The Fourier cosine series for $f(x)$ on the interval $[0, 2]$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\tfrac{n\pi x}{2} = \frac{3}{2} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\tfrac{n\pi}{2} - 1\right]\cos\tfrac{n\pi x}{2}$$

for $0 < x < 2$.

g is <u>even</u>

$$\int_{-L}^{L} g(x)\,dx = \int_{-L}^{0} g(x)\,dx + \int_{0}^{L} g(x)\,dx$$

$t = -x$

$$= -\int_{L}^{0} g(-t)\,dt + \int_{0}^{L} g(x)\,dx$$

$$= \int_{0}^{L} g(-t)\,dt + \int_{0}^{L} g(t)\,dt$$

$$= 2\int_{0}^{L} g(t)\,dt \,.$$

---

g <u>odd</u>

$$\int_{-L}^{L} g(x)\,dx = -\int_{L}^{0} g(-t)\,dt + \int_{0}^{L} g(x)\,dx$$

$$= \int_{0}^{L} g(-t)\,dt + \int_{0}^{L} g(t)\,dt$$

$$= -\int_{0}^{L} g(t)\,dt + \int_{0}^{L} g(t)\,dt = 0$$

Fourier Series of $f(x)$ on $[-L, L]$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \qquad -L \le x \le L$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

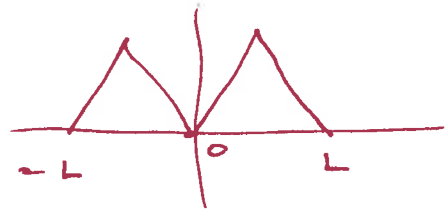
for $n \ge 1$.

Note: If f is even, have only cosine terms

If f is odd, have only sine terms

1. If $f(x)$ is even, $-L \le x \le L$

$$f(-x) = f(x)$$



So $f(x) \sin \frac{n\pi x}{L}$ is odd

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = 0, \quad n \ge 1$$

So

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx = \frac{1}{L}\int_0^L f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

---

2. If $f(x)$ is odd, $-L \le x \le L$

$$f(-x) = -f(x)$$

So $f(x) \cos \frac{n\pi x}{L}$ is odd

and

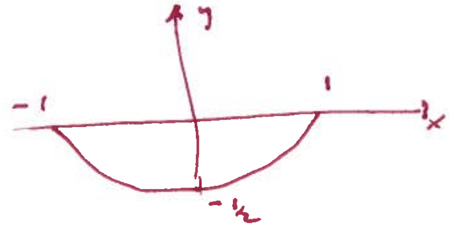
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = 0 \quad \forall n \ge 1$$

So

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Ex 1:



Find Fourier series of

$$f(x) = x^2 - \tfrac{1}{2}, \qquad -1 \le x \le 1 \qquad\qquad \underline{L=1}$$

$f(-x) = f(x)$ and f is $\underline{\text{even}}$, so

$$f(x) \cdot \sin n\pi x = \underline{\text{odd}} \quad \text{and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = 0 \qquad \forall\, n \ge 1$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\, dx = -\frac{1}{6}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x) \cos n\pi x\, dx = 2\int_0^1 (x^2 - \tfrac{1}{2}) \cos n\pi x\, dx = \frac{4}{n^2\pi^2}(-1)^n$$

So

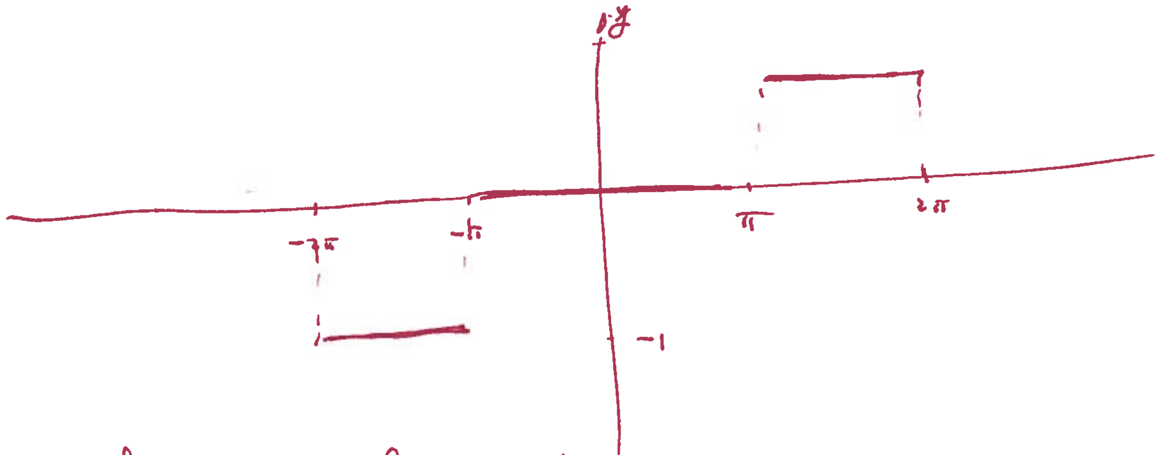
$$x^2 - \tfrac{1}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x, \qquad -1 \le x \le 1$$

and

$$x^2 - \tfrac{1}{2} = -\tfrac{1}{6} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x$$

Ex 2: Find Fourier Series f

$$f(x) = \begin{cases} -1 & -2\pi < x < -\pi \\ 0 & -\pi < x < \pi \\ 1 & \pi < x < 2\pi \end{cases} \qquad \underline{L = 2\pi}$$



$f(-x) = -f(x)$ and f is odd so

$f(x)\cos\frac{nx}{2}$ is odd

and

$$a_0 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\cos\frac{nx}{2}\,dx = 0 \quad \text{for } n \geq 1.$$

$$b_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\sin\frac{nx}{2}\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{f(x)\sin\frac{nx}{2}}_{\text{even}}\,dx$$

$$b_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \sin\frac{nx}{2}\,dx = -\frac{1}{\pi}\frac{2}{n}\cos\left(\frac{1}{2}x\right)\Big|_{\pi}^{2\pi} = -\frac{2}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

Math 300
Winter 2011

Math 300
Solutions to Midterm Examination I

Question 1. The neutron flux u in a sphere of uranium obeys the differential equation

$$\frac{\lambda}{3}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+(k-1)A\,u=0$$

for $0<r<a$, where λ is the effective distance traveled by a neutron between collisions. A is called the absorption cross section, and k is the number of neutrons produced by a collision during fission. In addition, the neutron flux at the boundary of the sphere is 0.

(a) Make the substitution

$$u=\frac{v}{r}\qquad\text{and}\qquad\mu^2=\frac{3(k-1)A}{\lambda}$$

and show that $v(r)$ satisfies $\dfrac{d^2v}{dr^2}+\mu^2\,v=0$, $0<r<a$.

(b) Find the general solution to the differential equation in part (a) and then find $u(r)$ that satisfies the boundary condition and boundedness condition:

$$u(a)=0\qquad\text{and}\qquad\lim_{r\to 0^+}|u(r)|\ \text{bounded}.$$

(c) Find the critical radius, that is, the smallest radius a for which the solution is not identically 0.

SOLUTION:

(a) Letting $u=v/r$, then

$$\frac{du}{dr}=\frac{1}{r}\frac{dv}{dr}-\frac{1}{r^2}v\qquad\text{and}\qquad r^2\frac{du}{dr}=r\frac{dv}{dr}-v,$$

so that

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=r\frac{d^2v}{dr^2}+\frac{dv}{dr}-\frac{dv}{dr}=r\frac{d^2v}{dr^2},$$

and the differential equation for $v(r)$ is

$$\frac{1}{r}\frac{d^2v}{dr^2}+\frac{\mu^2}{r}v=0,\qquad\text{that is,}\qquad\frac{d^2v}{dr^2}+\mu^2v=0$$

for $0<r<a$.

(b) The general solution to the differential equation in part (a) is

$$v(r)=c_1\cos\mu r+c_2\sin\mu r$$

for $0<r<a$, and the solution to the neutron flux equation is

$$u(r)=\frac{v(r)}{r}=c_1\frac{\cos\mu r}{r}+c_2\frac{\sin\mu r}{r}$$

for $0<r<a$. Applying the boundedness condition, since

$$\lim_{r\to 0^+}\frac{\sin\mu r}{r}=\mu\qquad\text{and}\qquad\lim_{r\to 0^+}\frac{\cos\mu r}{r}\ \text{doesn't exist},$$

then we need $c_1=0$, and the solution is

$$u(r)=c_2\frac{\sin\mu r}{r}$$

for $0<r<a$.

(c) Applying the boundary condition

$$u(a)=\frac{c_2}{a}\sin\mu a=0,$$

clearly, there is a nontrivial solution if and only if $\mu a=n\pi$ for some positive integer n. The critical radius is $a=\dfrac{\pi}{\mu}$.

In 2.4.1

Question 2. Show that

$$|\sin x| = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1}\cos nx$$

for $-\infty < x < \infty$.

Hint: Find the Fourier cosine series of the function $f(x) = \sin x$, $0 \le x \le \pi$, and then use Dirichlet's theorem.

SOLUTION: For $0 \le x \le \pi$, the Fourier cosine series is

$$\sin x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

for $n = 0$,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\, dx = -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}[-1-1] = \frac{2}{\pi}.$$

for $n = 1$,

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} 2\sin x\cos x\, dx \\ &= \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx = -\frac{1}{2\pi}\cos 2x\Big|_0^{\pi} \\ &= -\frac{1}{2\pi}[1-1] = 0. \end{aligned}$$

and for $n > 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx \\ &= -\frac{1}{\pi(n+1)}\cos(n+1)x\Big|_0^{\pi} + \frac{1}{\pi(n-1)}\cos(n-1)x\Big|_0^{\pi} \\ &= -\frac{1}{\pi}\left[\frac{(-1)^{n+1}-1}{n+1}\right] + \frac{1}{\pi}\left[\frac{(-1)^{n-1}-1}{n-1}\right] \\ &= -\frac{2}{\pi}\frac{1+(-1)^n}{n^2-1}. \end{aligned}$$

The Fourier cosine series for $\sin x$ on the interval $-\pi \le x \le \pi$ is therefore

$$\sin x \sim \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}\cos nx.$$

This is also the Fourier series for the even extension of $\sin x$, that is, $|\sin x|$, on the interval $-\pi \le x \le \pi$, so that

$$|\sin x| \sim \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}\cos nx,$$

and since this is a continuous piecewise smooth function on $-\pi \le x \le \pi$ such that $|\sin(-\pi)| = |\sin\pi| = 0$, then by Dirichlet's theorem the Fourier series converges to $|\sin x|$ for all $x \in [-\pi, \pi]$, and

$$|\sin x| = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}\cos nx$$

for all $x \in [-\pi,\pi]$. However, $|\sin x|$ is a continuous periodic function on $-\infty < x < \infty$, so the Fourier series converges to $|\sin x|$ for all x, $-\infty < x < \infty$.

Question 3. Solve Laplace's equation in the square $0 \le x \le \pi$, $0 \le y \le \pi$ with the boundary conditions given below

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le \pi, \quad 0 \le y \le \pi$$

$$u(0, y) = 0, \qquad 0 \le y \le \pi$$

$$u(\pi, y) = 0, \qquad 0 \le y \le \pi$$

$$u(x, 0) = 0, \qquad 0 \le x \le \pi$$

$$u(x, \pi) = 1, \qquad 0 \le x \le \pi.$$

SOLUTION: We use separation of variables and assume a solution to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

of the form $u(x, y) = X(x) \cdot Y(y)$.

Separating variables we have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(t)}{Y(t)} = -\lambda.$$

and we obtain the two ordinary differential equations

$$X'' + \lambda X = 0 \quad 0 \le x \le \pi \qquad Y'' - \lambda Y = 0, \quad 0 \le y \le \pi$$

$$X(0) = 0 \qquad\qquad Y(0) = 0$$

$$X(\pi) = 0.$$

Solving the complete boundary value problem for X, the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2 \qquad \text{and} \qquad X_n(x) = \sin nx$$

for $n \ge 1$.

The corresponding problem for Y is

$$Y'' - n^2 Y = 0$$

$$Y(0) = 0$$

with solutions

$$Y_n(y) = \sinh ny$$

for $n \ge 1$.

From the superposition principle, we write

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh ny \sin nx,$$

and setting $y = \pi$, we need

$$1 = u(x, \pi) = \sum_{n=1}^{\infty} b_n \sinh n\pi \sin nx,$$

and from the orthogonality of the eigenfunctions.

$$b_n \sinh n\pi = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = -\frac{2}{n\pi} \cos nx \Big|_0^{\pi} = -\frac{2}{n\pi} [(-1)^n - 1],$$

so that

$$u(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n \sinh n\pi} \sin nx \sinh ny$$

for $0 \le x \le \pi$, $0 \le y \le \pi$.

Lecture 12:

Fourier's Theorem: **Fourier-series** of $f \in PWS[-L, L]$

$$FS(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

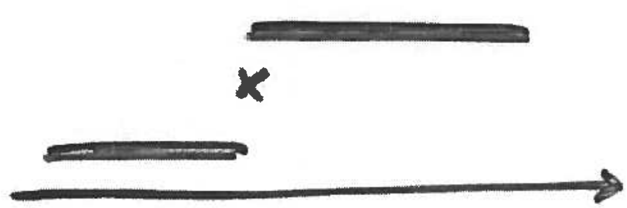
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

if

(i) $f(x)$ continuous: $\Rightarrow$ $f(x) = FS(x)$ for all $x \in [-L, L]$

if

(ii) $f(x)$ jump: $\dfrac{f(x+) + f(x-)}{2} = FS(x)$ for $-L < x < L$



(iii) At the endpoints $x_0 = \pm L$, Fourier series converges to

$$\frac{f(-L^+) + f(L^-)}{2}$$

Fourier - cosine - series

$f \in PWS[0, L]$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L} \int_0^L f(x)\,dx$$

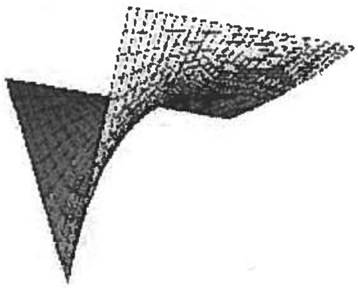
$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Fourier - sine - series

$f \in PWS[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$



Math 300 Winter 2009
Advanced Boundary Value Problems I

Dirichlet's Theorem

Wednesday February 2, 2011

Department of Mathematical and Statistical Sciences
University of Alberta

The following theorem on the convergence of Fourier series in known as **Dirichlet's Theorem**, or **Fourier's Theorem**.

Theorem. Let $f(x)$ be piecewise smooth on the interval $[-L, L]$, the Fourier series

$$\mathcal{F}\,S(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\tfrac{n\pi x}{L} + b_n \sin\tfrac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\tfrac{n\pi t}{L}\,dt, \quad n \ge 1$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\tfrac{n\pi t}{L}\,dt, \quad n \ge 1$$

has the following properties:

(i) If $f(x)$ is continuous at x_0, where $-L < x_0 < L$, then

$$f(x_0) = \mathcal{F}\,S(x_0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\tfrac{n\pi x_0}{L} + b_n \sin\tfrac{n\pi x_0}{L}\right),$$

that is, the Fourier series converges to $f(x_0)$.

(ii) If $f(x)$ has a jump discontinuity at x_0, where $-L < x_0 < L$, then

$$\frac{f(x_0^+) + f(x_0^-)}{2} = \mathcal{F}\,S(x_0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\tfrac{n\pi x_0}{L} + b_n \sin\tfrac{n\pi x_0}{L}\right),$$

that is, the Fourier series converges to the **average** or the **mean** of the jump.

(iii) At the endpoints $x_0 = \pm L$, the Fourier series converges to

$$\frac{f(-L^+) + f(L^-)}{2}.$$

Usually we write

$$f(x) \sim \mathcal{F}\,S(x_0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\tfrac{n\pi x_0}{L} + b_n \sin\tfrac{n\pi x_0}{L}\right).$$

and say that $f(x)$ is *represented by its Fourier series* on the interval $[-L, L]$.

Note: The Fourier series $\mathcal{F}S(x)$ is periodic with period $2L$, and converges for **all** real numbers x.

For each real number x_0 it converges to the $2L$-periodic extension $\widehat{f}(x_0)$ of f whenever $\widehat{f}$ is continuous at x_0, and to

$$\frac{\widehat{f}(x_0^+) + \widehat{f}(x_0^-)}{2}$$

whenever $\widehat{f}$ has a jump discontinuity at x_0.

This allows us to sketch the graph of the Fourier series $\mathcal{F}S(x)$ once we know what the graph of the function $f(x)$ is, without computing the Fourier series coefficients a_0, a_n and b_n for $n \geq 1$.

Exercise. Sketch the graph of the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -L \leq x < 0 \\ e^{-x} & \text{for} \quad 0 < x < \dfrac{L}{2} \\ 0 & \text{for} \quad \dfrac{L}{2} < x \leq L. \end{cases}$$

Note that the function is undefined for $x = 0$ and $x = \dfrac{L}{2}$, and this does not affect the integrals when we compute the Fourier coefficients. We can sketch the graph of the Fourier series using Dirichlet's theorem. and then go back and redefine the function at the points of discontinuity so that the Fourier series converges to the function for all $x \in [-L, L]$.

Question 8. [p 126, #3.4.8]

Consider the boundary value – initial value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0; \qquad \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0; \qquad u(x,0) = f(x), \quad 0 < x < L.$$

Solve this problem by looking for a solution as a Fourier cosine series. Assume that u and $\dfrac{\partial u}{\partial x}$ are continuous and $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial u}{\partial t}$ are piecewise smooth. Justify all differentiations of infinite series.

SOLUTION: We assume a solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos\frac{n\pi x}{L}$$

and assuming all derivatives are continuous, we have

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=0}^{\infty} a_n(t)\left(\frac{n\pi}{L}\right)^2 \cos\frac{n\pi x}{L}$$

and since $u(x,t)$ satisfies the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

then we have

$$\sum_{n=0}^{\infty} a_n'(t)\cos\frac{n\pi x}{L} = -k\sum_{n=0}^{\infty} a_n(t)\left(\frac{n\pi}{L}\right)^2 \cos\frac{n\pi x}{L}.$$

Collecting terms that multiply $\cos\frac{n\pi x}{L}$ for $n \geq 0$ for $n \geq 1$, and using the fact that these trigonometric functions are linearly independent (they are orthogonal on the interval $[0, L]$), then we get

$$a_n'(t) = -ka_n(t)\left(\frac{n\pi}{L}\right)^2,$$

and we can solve these first order linear ordinary differential equations for $a_n(t)$ to get

$$a_n(t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt},$$

and the solution $u(x,t)$ becomes

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\cos\frac{n\pi x}{L}.$$

Differentiating this with respect to x, we get

$$\frac{\partial u}{\partial x}(x,t) = -\sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L},$$

and setting $x = 0$, we get

$$0 = \frac{\partial u}{\partial x}(0,t),$$

and the first boundary condition is satisfied.

The solution is now

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \cos \frac{n\pi x}{L},$$

and we note that the second boundary condition $\dfrac{\partial u}{\partial x}(L,t) = 0$ is also satisfied, so we only need to find the constants A_n to satisfy the initial condition $u(x,0) = f(x)$.

Setting $t = 0$ in the above expression for $u(x,t)$, we have

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L},$$

and the A_n are the Fourier cosine series coefficients of $f(x)$, so that

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \quad n \geq 1$$

and for $n = 0$,

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx.$$

Question 9. [p 85, #2.5.1 (b),(g)] Solve Laplace's equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the following boundary conditions:

(a) $\dfrac{\partial u}{\partial x}(0,y) = g(y), \qquad \dfrac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = 0, \qquad u(x,H) = 0$

(b) $\dfrac{\partial u}{\partial x}(0,y) = 0, \qquad \dfrac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = \begin{cases} 1 & \text{for } \ 0 < x < L/2, \\ 0 & \text{for } \ L/2 < x < L \end{cases} \qquad \dfrac{\partial u}{\partial y}(x,H) = 0$

SOLUTION:

(a) We assume a solution of the form $u(x,y) = X(x) \cdot Y(y)$, and substituting this into Laplace's equation we have

$$X''(x) \cdot Y(y) + X(x) \cdot Y''(y) = 0,$$

and

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda,$$

so we get two ordinary differential equations

$$X''(x) - \lambda X(x) = 0 \qquad \text{and} \qquad Y''(y) + \lambda Y(y) = 0.$$

We can satisfy the (homogeneous) boundary conditions by requiring that

$$Y(0) = 0, \quad Y(H) = 0 \qquad \text{and} \qquad X'(L) = 0.$$

Therefore X and Y satisfy the boundary value problems

$$\begin{array}{ll} X''(x) - \lambda X(x) = 0, \quad 0 \leq x \leq L & Y''(y) + \lambda Y(y) = 0, \quad 0 \leq y \leq H \\ X'(L) = 0 & Y(0) = 0 \\ & Y(H) = 0. \end{array}$$

We solve the complete (Dirichlet) boundary value problem for Y first, the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2$$

with corresponding eigenfunctions

$$Y_n(y) = \sin \tfrac{n\pi}{H} y$$

for $n \geq 1$.