Lecture 10

2 Fourier Series

(2.1) Recall : Linear Algebra

Vector space $\mathbb{R}^n$

basis vectors $e_1 = (1, 0, \dots, 0)$

$e_2 = (0, 1, 0, \dots, 0)$

orthogonal basis of $\mathbb{R}^n$ $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$

orthogonal: $\langle e_i, e_j \rangle = e_i^T e_j = \delta_{ij}$

basis representation: $a \in \mathbb{R}^n$

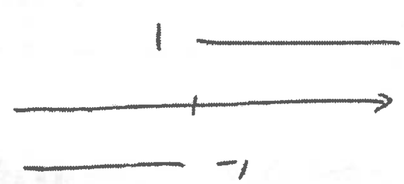
$$a = \sum_{i=1}^{n} a_i e_i \qquad a_j = \langle a \cdot e_j \rangle$$

(2.2) Piecewise smooth functions

Def 1: A function $f(x)$ has a jump discontinuity at x_0, if the right and left sided limits exist and

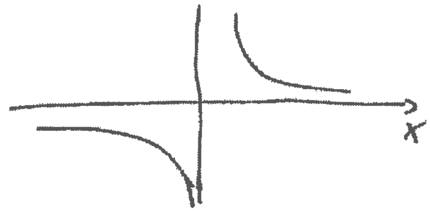
$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$$

Example 1 Heaviside function $H(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0. \end{cases}$



<u>Example 2</u> Not a jump discontinuity at $x_0 = 0$:

$f(x) = \frac{1}{x}$



$\lim_{x \to 0^-} f(x) = -\infty$ $\qquad$ $\lim_{x \to 0+} f(x) = +\infty$.

Def $\quad f \in PWC$ iff

<u>Def 2</u>: A function $f(x)$ on $[a, b]$ is called piecewise smooth, if f and f' has at most a finite number of jump discontinuities.

e.g.

The set of all piecewise smooth functions on $[a, b]$ is called $PWS\,[a, b]$.

$PWS[a, b]$ is a <u>vectorspace</u> !

Def: Inner Product on vector space V (real):

For $u, v \in V$

$\langle u, v \rangle$.

Satisfying

(i) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

(ii). $\langle u, v \rangle = \langle v, u \rangle$

(iii) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(iv). $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$

Note: if $f, g \in C[a,b]$ then

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

is an inner product on $C[a,b]$. But

- if $f, g \in PWC[a,b]$ then $\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt = 0$

$\not\Rightarrow$ $f(t) = 0$ for all $t \in [a,b]$.

So we identify those functions which differ on a finite number of points. With this identification, $\langle \cdot, \cdot \rangle$ is an inner product on $PWC[a,b]$ and $PWS[a,b]$.

Note: For $C[a,b]$,

Only (i) has to be shown, i.e.

$$\langle u, u\rangle = 0 \text{ iff } u = 0.$$

Suppose $f \in C[a,b]$ and $f(x) = 0$ for all $x \in [a,b]$

then $\int_a^b f(x)\, f(x)\, dx = \int_a^b 0\, dx = 0$

Conversely, suppose $f \in C[a,b]$ and $\langle f, f\rangle = 0$.

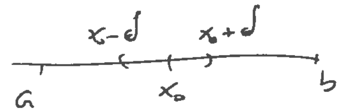
if $f \neq 0$, then $\exists\ x_0 \in [a,b] \ni f(x_0) \neq 0$

So given $\epsilon = \frac{1}{2}|f(x_0)| > 0$ there is a $\delta > 0 \ni$

assume $f(x_0) > 0$.

$$|f(x) - f(x_0)| < \epsilon = \tfrac{1}{2}|f(x_0)|$$

for all x with $|x - x_0| < \delta$.



So

$$-\tfrac{1}{2}|f(x_0)| \leq f(x) - f(x_0) \leq \tfrac{1}{2}|f(x_0)|$$

so

$$f(x_0) - \tfrac{1}{2}f(x_0) \leq f(x) \leq f(x_0) + \tfrac{1}{2}f(x_0)$$

$\forall\ x \in (x_0 - \delta, x_0 + \delta)$

i.e. $0 < \frac{1}{2}f(x_0) \leq f(x) \leq \frac{3}{2}f(x_0) \quad \forall\ x \in (x_0-\delta, x_0+\delta)$

then $\int_a^b f(x)^2\, dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)^2\, dx \geq \frac{1}{4}\int_{x_0-\delta}^{x_0+\delta} f(x_0)^2\, dx = \frac{\delta}{2}\cdot f(x_0) > 0$

※ since we assumed $\int_a^b f(x)^2\, dx = 0$

∎

inner product in PWS:

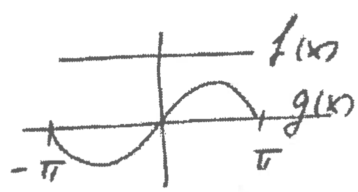
$$f, g \in PWS[a, b]$$

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

Orthogonality in PWS: $\langle f, g \rangle = 0$

examples: $f(x) = 3$, $g(x) = \sin x$ are orthogonal in $PWS[-\pi, \pi]$:

$$\langle 3, \sin x \rangle = \int_{-\pi}^{\pi} 3 \sin x\, dx = -3 \left[\cos \pi - \cos(-\pi)\right] = 0$$



$\cos\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal in $PWS[-L, L]$ and in $PWS[0, L]$ for $n \neq m$.

Let $\mathcal{C} := \left\{ \cos\left(\frac{n\pi x}{L}\right),\ n = 0, 1, 2, \ldots \right\}$

$\mathcal{S} := \left\{ \sin\left(\frac{m\pi x}{L}\right),\ m = 1, 2, 3, \ldots \right\}$

Theorem: $\mathcal{S} \cup \mathcal{C}$ forms an orthogonal basis of $PWS[-L, L]$.

$$\langle 1, \sin(\tfrac{n\pi x}{L}) \rangle = 0$$

$$\langle \sin(\tfrac{n\pi x}{L}), \sin(\tfrac{m\pi x}{L}) \rangle = 0 \qquad n \neq m$$

$$\langle \sin(\tfrac{n\pi x}{L}), \cos(\tfrac{m\pi x}{L}) \rangle = 0$$

etc.

S ∪ C orthogonal basis $\Rightarrow$ each function $f \in PWS[-L, L]$ must have a basis representation:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

<u>Fourier-series</u>.

We find a_0, a_n and b_n from orthogonality.

Multiply $f(x)$ by $\cos(\frac{m\pi x}{L})$ and integrate, $m > 0$:

$$\langle \cos(\tfrac{m\pi x}{L}), f(x) \rangle$$

$$= \langle \cos(\tfrac{m\pi x}{L}), a_0 \rangle + \sum_{n=1}^{\infty} a_n \langle \cos(\tfrac{m\pi x}{L}), \cos(\tfrac{n\pi x}{L}) \rangle$$

$$+ b_n \langle \cos(\tfrac{m\pi x}{L}), \sin(\tfrac{n\pi x}{L}) \rangle$$

$$= a_m \left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle$$

$$\left\langle \cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \cos^2\left(\frac{m\pi x}{L}\right) dx$$

$$= \int_{-L}^{L} \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2m\pi x}{L}\right) dx$$

$$= L$$

Hence for $m > 0$: $\quad a_m = \frac{1}{L}\left\langle \cos\left(\frac{m\pi x}{L}\right), f(x)\right\rangle$

or $\quad \boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) dx} \quad m > 0.$

Similarity: $\quad \boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right) dx} \quad m > 0$

a_0: Integrate $f(x)$:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} a_0\,dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx$$

$$+ b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= 2L\, a_0$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx.$$

<u>Theorem</u>: Let $f(x)$ be piecewise smooth on $[-L, L]$.

The Fourierseries

$$FS(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

has the following properties:

(i) Whenever $f(x)$ is continuous we have

$$f(x) = FS(x)$$

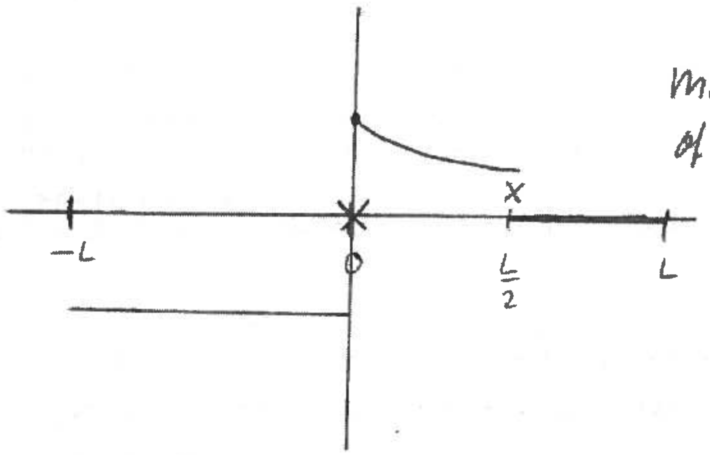
(ii) A jump-discontinuities x_0 we have

$$\frac{f(x^+) + f(x^-)}{2} = FS(x)$$

mean value of the jump.

Sketch the Fourier series of

$$f(x) = \begin{cases} -1 & -L \le x \le 0 \\ e^{-x} & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases}$$

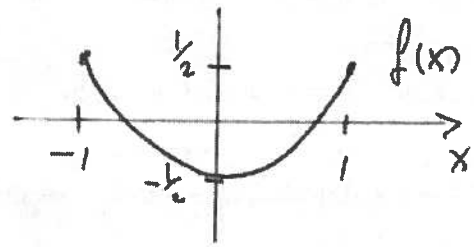


mark the mean value of a jump by an X.

Lecture 10:

Example 1: Find the Fourier series of $L = 1$

$f(x) = x^2 - \frac{1}{2}$ on the interval $[-1, 1]$.



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-1}^{1}\left(x^2 - \tfrac{1}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{2}x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2} - \left(-\frac{1}{3} + \frac{1}{2}\right)\right]$$

$$= -\frac{1}{6}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1}\left(x^2 - \tfrac{1}{2}\right)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{1} x^2\cos(n\pi x)\,dx - \frac{1}{2}\int_{-1}^{1}\cos(n\pi x)\,dx$$

$$= x^2\frac{\sin(n\pi x)}{n\pi}\Bigg]_{-1}^{1} - \int_{-1}^{1} 2x\frac{\sin(n\pi x)}{n\pi}\,dx - \frac{1}{2}\left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{1}$$

$$= 0 \quad + 2\left[x\frac{\cos(n\pi x)}{(n\pi)^2}\right]_{-1}^{1} - \int_{-1}^{1} 2\frac{\cos(n\pi x)}{(n\pi)^2}\,dx \quad - 0$$

$$= 2\left(\frac{\cos(n\pi)}{(n\pi)^2} + \frac{\cos(-n\pi)}{(n\pi)^2}\right) + 2\left.\frac{\sin(n\pi x)}{(n\pi)^3}\right]_{-1}^{1}$$

$$= 4\,\frac{\cos(n\pi)}{(n\pi)^2}$$

$$a_1 = \frac{-4}{\pi^2}, \quad a_2 = \frac{+4}{4\pi^2} = \frac{1}{\pi^2}, \quad a_3 = \frac{-4}{9\pi^2}$$

$$a_4 = \frac{4}{16\pi^2} = \frac{1}{4\pi^2}, \quad a_5 = -\frac{4}{25\pi^2}$$

$$b_n = \int_{-1}^{1} (x^2 - 1/2)\sin(n\pi x)\,dx$$

$$= \int_0^1 (x^2 - 1/2)\sin(n\pi x)\,dx + \int_{-1}^{0} (x^2 - 1/2)\sin(n\pi x)\,dx$$

$$= \int_0^1 (x^2 - 1/2)\sin(n\pi x)\,dx - \int_0^1 (x^2 - 1/2)\sin(-n\pi x)(-dx)$$

$= 0$ (short: $(x^2 - 1/2)\sin(n\pi x)$ is <u>odd</u>.)

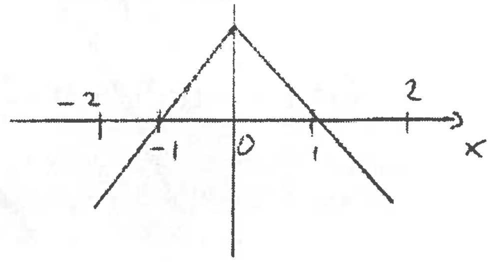
$\Rightarrow$ Fourier series:

$$f(x) = -\frac{1}{6} - \frac{4}{\pi^2}\cos(\pi x) + \frac{1}{\pi^2}\cos(2\pi x) - \frac{4}{9\pi^2}\cos(3\pi x)$$

$$+ \frac{4}{4\pi^2}\cos(4\pi x) - \frac{4}{25\pi^2}\cos(5\pi x) + \cdots$$

Example 2 Find the Fourier-series of

$$f(x) = \begin{cases} 1+x, & -2 < x < 0 \\ 1-x, & 0 < x \leq 2 \end{cases}$$



1) $f(-x) = f(x) \Rightarrow f(x)$ is even

$\Rightarrow f(x) \sin\left(\frac{n\pi x}{L}\right)$ is odd

i.e. $f(-x) \sin\left(\frac{n\pi(-x)}{L}\right) = -f(x) \sin\left(\frac{n\pi x}{L}\right)$

hence

$$b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin\left(\frac{n\pi x}{2}\right) dx = 0$$

2)

$$a_0 = \frac{1}{2 \cdot 2} \int_{-2}^{2} f(x)\, dx = \frac{1}{4} \int_{-2}^{0} (1+x)\, dx + \frac{1}{4} \int_{0}^{2} (1-x)\, dx$$

$$= \frac{1}{4} \left(\left[x + \frac{x^2}{2}\right]_{-2}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{2} \right)$$

$$= \frac{1}{4} \left(+2 - 2 + 2 - 2 \right) = 0$$

3)

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2} \int_{-2}^{0} (1+x) \cos\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2} \int_{0}^{2} (1-x) \cos\left(\frac{n\pi x}{2}\right) dx$$

(~~Proof~~ better: $f(x) \cos\left(\frac{n\pi x}{2}\right) = f(-x) \cos\left(\frac{n\pi(-x)}{2}\right)$

$f(x)\cos\left(\frac{n\pi x}{2}\right)$ is even.

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_{0}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{2} (1-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{2} \cos\left(\frac{n\pi x}{2}\right)dx - \int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big]_{0}^{2} - \left[x\,\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{0}^{2}$$

$$+ \int_{0}^{2} \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{-4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)\Big]_{0}^{2}$$

$$= \frac{-4}{(n\pi)^2}\left(\cos(n\pi) - 1\right)$$

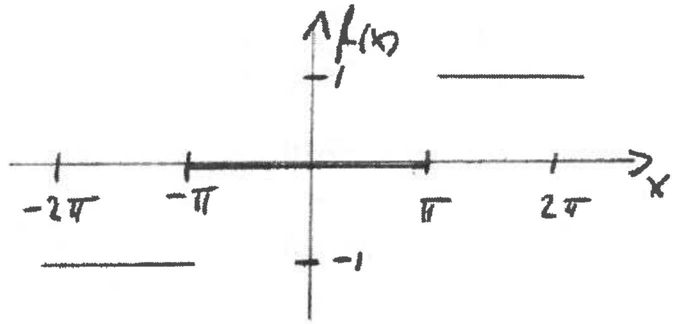
$$= \frac{4}{(n\pi)^2}\left(1 - (-1)^n\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{2}\right)$$

$$= \frac{8}{\pi^2}\cos\left(\frac{\pi x}{2}\right) + \frac{8}{9\pi^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{8}{25\pi^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots$$

Example 3 Find the Fourier-series of

$$f(x) = \begin{cases} -1 & , \quad -2\pi < x < -\pi \\ 0 & , \quad -\pi < x < \pi \\ 1 & , \quad \pi < x < 2\pi \end{cases}$$



1) $L = 2\pi$, $f(-x) = -f(x)$ odd function

$$\Rightarrow \quad f(x)\cos\left(\frac{n\pi x}{2\pi}\right) = f(x)\cos\left(\frac{n}{2}x\right) \quad \text{also odd}$$

$$\Rightarrow \quad a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\cos\left(\frac{n}{2}x\right)dx = 0$$

$$a_0 = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f(x)\,dx = 0$$

2) $$b_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\sin\left(\frac{n}{2}x\right)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin\left(\frac{n}{2}x\right)dx$$

because $f(x)\sin\left(\frac{n}{2}x\right)$ is even.

$$= \frac{1}{\pi}\int_{\pi}^{2\pi}\sin\left(\frac{n}{2}x\right)dx = -\frac{1}{\pi}\frac{2}{n}\cos\left(\frac{n}{2}x\right)\Big]_{\pi}^{2\pi}$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n}{2}\pi\right)\right)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos\left(\frac{n}{2}\pi\right) - \cos(n\pi)\right)\sin\left(\frac{n}{2}x\right)$$

$$= \frac{2}{\pi}\left(0-(-1)\right)\sin\left(x/2\right) + \frac{1}{\pi}\left((-1)-1\right)\sin(x)$$

$$+ \frac{2}{3\pi}\left(0-(-1)\right)\sin\left(\frac{3}{2}x\right) + \frac{1}{2\pi}(1-1)\sin(2x) + \cdots$$

$$= \frac{1}{\pi}\left(2\sin\left(x/2\right) - 2\sin x + \frac{2}{3}\sin\left(\frac{3}{2}x\right) + \cdots\right)$$