Midterm I:

June 6 9:00 - 9:50 in CAB 243

(No electronics, bring OneCard)

3 Problems:
- Separation of Variables
- Fourier Series
- PDEs

Start Fourier Series on Friday
May 25 — June 4 } Chap 2

Essential Steps for Separation:

Running Example: Heat Eqn on rod of length L

x=0 x=L

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{PDE}$$

$u(0,t) = 0$ for all $t>0$
$u(L,t) = 0$ for all $t>0$ } B.C.

$\boxed{u(x,0) = f(x), \quad 0 \le x \le L}$ } I.C.

Boundary Value - Initial Value Problem
(BV-IV)P

1. Separation: Only works when
B.C. are linear and homogeneous
P.D.E. is linear and homogeneous

Assume a soln of the form
$\boxed{u(x,t) = \varphi(x) \cdot G(t)}$

2. Substitute this into the PDE and the homogeneous part of boundary and initial conditions and derive two ordinary differential eqns for $\varphi(x)$ and $G(t)$

$$\varphi(x)\, G'(t) = k\, \varphi''(x) \cdot G(t)$$

$$\frac{\varphi''(x)}{\varphi(x)} = \frac{G'(t)}{k\, G(t)} \qquad (*)$$

$\uparrow$ depends only on x, not t.

$\uparrow$ depends only on t, not on x

Since x and t are independent variables

So from $(*)$ $\quad \dfrac{\varphi''(x)}{\varphi(x)} = -\lambda = \dfrac{G'(t)}{k\, G(t)}$

Ord. diff eqns.

x-eqn:

$$\varphi''(x) + \lambda \varphi(x) = 0, \quad 0 \le x \le L$$

$$\varphi(0) = \varphi(L) = 0$$

t-eqn:

$$G'(t) + \lambda k\, G(t) = 0, \quad t > 0$$

3. Identify the complete problem of these two (the one with the correct number of side conditions so we can solve it)

e.g. we get a nontrivial solution to the x-eqn for those values of λ such that

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \leftarrow \text{eigenvalues}$$

with corresponding eigenfunctions

$$\varphi_n(x) = \sin\left(\frac{n\pi}{L}x\right) \leftarrow \text{eigenfunctions}$$

4. Use those values of λ to solve the time problem:

$$G_n'(t) + k\lambda_n G_n(t) = 0$$

$$G_n(t) = z_n e^{-k\lambda_n t} \quad t > 0$$

5. Use superposition principle

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) \quad 0 \le x \le L,\ t > 0}$$

Satisfies PDE + the BCs but not the initial condition

4. Use these values of λ_n to solve the time problem:

$$G_n'(t) + k\lambda_n G_n(t) = 0$$

$$G_n(t) = Z_n e^{-k\lambda_n t} \qquad t>0$$

5. Use superposition principle

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) \qquad 0 \le x \le L,\ t>0}$$

Satisfies PDE & the B.C.s but not the initial condition

6. Use the nonhomogeneous side condition by setting $t=0$ to get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}\right)x$$

7. Find the Fourier sine series of $f(x)$ to get the constants a_n

$t>0$